The convex structure of the set of positive approximants for a given operator

By RICHARD BOULDIN in Athens (Georgia, U.S.A.)

§ 1. Introduction

In [6], P. R. HALMOS showed that any (bounded linear) operator T has a positive approximant, denoted by P_0. This means that P_0 is a nonnegative operator and the norm $\|T-P_0\|$ is the same as the distance from T to the set of nonnegative operators. Other basic facts were collected in [6] and in [2]. Halmos asked for the extreme points of the convex set of positive approximants, denoted by $\mathscr{P}(T)$, and for a characterization of those T for which $\mathscr{P}(T)$ is a singleton set. This paper characterizes a normal operator T for which $\mathscr{P}(T)$ is q-dimensional and constructs some extreme points of that set. (For dimension of a convex set see pp. 7—9 of [8].)

In [3] we studied the set of positive near-approximants of T, denoted $\mathscr{P}'(T)$, where a positive near-approximant is a best approximation for T using the new norm

$$|||T|||^2 = \|B^2+C^2\|$$

with $T=B+iC$, $B=B^*$, $C=C^*$. The distance from T to the nonnegative operators is the same whether it is computed with the new norm or with the operator norm. This distance is denoted by $\delta(T)$ and is referred to as the *modulus of positivity*. Recall from [3] that the new norm is between the operator norm and the numerical radius. We use [5] as a source for many terms and facts that we shall not explain.

§ 2. Preliminaries

Frequently in the study of a convex set the dimension of the convex set is apparent. Then generally the investigation turns to the subtler question of determinig the extreme points of the convex set. Moreover, if the nonempty convex set is a compact subset of some locally convex topological vector space then the Krein—Milman theorem implies that the closed convex hull of the extreme points of the convex set is the convex set. In the case that the dimension of either $\mathscr{P}(T)$ or $\mathscr{P}'(T)$ is finite then the following theorem should inspire some interest in the extreme points.

2.1. Theorem. *Both of the convex sets $\mathscr{P}(T)$ and $\mathscr{P}'(T)$ are the closed convex hull of their extreme points.*

Proof. Clearly the set of bounded operators on H is a locally convex topological vector space with the weak operator topology and so it suffices to show that each of the two sets is compact in the weak operator topology. Since any ball of bounded operators is compact in the weak operator topology (see problem 6 p. 512 [4]) and since both sets $\mathscr{P}(T)$ and $\mathscr{P}'(T)$ are obviously contained in such balls, it would suffice to show that both sets are closed in the weak operator topology. We define two functions φ and ψ on the bounded operators of H by the formulas

$$\varphi(S) = \|T-S\|, \quad \psi(S) = |||T-S|||.$$

Let $\mathscr{R}=\{S:\varphi(S)>\delta(T)\}$ and $\mathscr{S}=\{S:\psi(S)>\delta(T)\}$. Since the nonnegative operators are obviously closed in the weak operator topology it would suffice to prove that $\mathscr{R}$ and $\mathscr{S}$ are open in that topology. In order to prove the last assertion it would suffice to show that φ and ψ are lower semicontinuous and it is clear that φ can be written as the supremum of functions which are obviously continuous with respect to the weak operator topology. It follows that $\mathscr{P}(T)$ is compact in the weak operator topology and the same conclusion will follow for $\mathscr{P}'(T)$ once we show that ψ is lower semicontinuous. The last property can be deduced from the following formula from Theorem 3.1 of [3]:

$$|||T||| = \frac{1}{2}\|T^*T+TT^*\|^{1/2} = \frac{1}{2}[w(T^*T+TT^*)]^{1/2}.$$

Any consideration of the convex structure of either $\mathscr{P}(T)$ or $\mathscr{P}'(T)$ will require the following theorem from [6].

2.2. Theorem. (Halmos) *If $T=B+iC$ with $B=B^*$, $C=C^*$ then*

$$\inf\{\|T-P\|:P\geqq 0\} = \inf\{r:B+(r^2-C^2)^{1/2}\geqq 0\}.$$

If the above quantity is denoted by δ then $P_0=B+(\delta^2-C^2)^{1/2}$ is a positive approximant for T.

§ 3. The main theorem

The main theorem of this paper will be proved with a sequence of lemmas in the next section. In this section we state the result.

3.1. Theorem. *Let $\mathscr{P}'(T)$ denote the convex set of positive near-approximants for the normal operator T and let*

$$H_0 = (P_0H)^- \cap ((\delta^2-C^2)H)^-.$$

If p is the dimension of H_0 then

$$\dim \mathscr{P}'(T) = p^2.$$

Here all infinite cardinal numbers are identified.

After this theorem is proved the techniques will be extended to obtain the same result for $\mathscr{P}(T)$. Then by restricting the generality we shall construct extreme points for both sets. However, we should note that a positive near-approximant is not necessarily a positive approximant. For example, the positive part of the real part of $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is a positive near-approximant although the Halmos positive approximant P_0 is the unique positive approximant.

In the sequel we shall repeatedly need the following result from [3].

3.2. Lemma. *If $R_1, \dots, R_n$ are commuting nonnegative operators on H then there is a nonnegative operator R such that*

(i) *$RR_j = R_jR$ for every j,*
(ii) *$R \leqq R_j$ for every j,*
(iii) *$(RH)^- = \bigcap \{(R_jH)^- : j=1, \dots, n\}$.*

§ 4. Proof of the main theorem

Notations. By A_0 we shall denote the positive operator constructed from P_0 and $2(\delta^2 - C^2)^{1/2}$ by appeal to Lemma 3.2 with T normal. Thus A_0 is dominated by P_0 and $2(\delta^2 - C^2)^{1/2}$; A_0 commutes with both operators and $(A_0H)^-$ is H_0.

4.1. Lemma. *If A is a positive operator such that*

$$0 \leqq A \leqq A_0$$

and A commutes with $(\delta^2 - C^2)^{1/2}$ then $P_0 - A$ is a positive near-approximant for T.

Proof. In view of the given inequality we have

$$0 \leqq A \leqq P_0 \quad \text{and} \quad A \leqq 2(\delta^2 - C^2)^{1/2}.$$

Thus we have the following inequality

$$-(\delta^2 - C^2)^{1/2} \leqq A - (\delta^2 - C^2)^{1/2} \leqq (\delta^2 - C^2)^{1/2}$$

and consequently we have

$$(\delta^2 - C^2)^{1/2} \pm \big(A - (\delta^2 - C^2)^{1/2}\big) \geqq 0.$$

Since A commutes with $(\delta^2 - C^2)^{1/2}$ the above two inequalities imply that

$$(\delta^2 - C^2) - \big(A - (\delta^2 - C^2)^{1/2}\big)^2 \geqq 0,$$

i.e.

$$\delta^2 \geqq \big(A - (\delta^2 - C^2)^{1/2}\big)^2 + C^2.$$

It now follows that

$$\delta^2 \geqq \|(A-(\delta^2-C^2)^{1/2})^2+C^2\| = \||A-(\delta^2-C^2)^{1/2}+iC|\|^2 =$$
$$= \||A+T-P_0|\|^2 = \||T-(P_0-A)|\|^2$$

and so P_0-A is a positive near-approximant for T.

4.2. Lemma. *Let $E(\cdot)$ be the spectral measure for a self adjoint operator A. If $E([a, b))\neq 0$ then either* (i) *or* (ii) *below holds:*

(i) $E([a, c))\neq 0$ and $E([c, b))\neq 0$ for *some* $c\in(a, b)$
(ii) $\sigma(A|E([a, b))H)=\{e\}$ for some $e\in[a, b)$.

Proof. Take two strictly monotone sequences, say $\{a_k: k=0, 1, \dots\}$ and $\{b_j: j=0, 1, \dots\}$, such that $a_0=(a+b)/2=b_0$, $a_k\to a$, $b_j\to b$ and $|a_k-a_{k+1}|<\frac{1}{2}$, $|b_j-b_{j+1}|<\frac{1}{2}$ for $j, k=0, 1, \dots$. Since

$$(*) \qquad [a, b) = \{a\}\cup\bigcup_{k=0}^{\infty}[a_{k+1}, a_k)\cup\bigcup_{j=0}^{\infty}[b_j, b_{j+1}),$$

at least one of the sets on the right of $(*)$ has nonzero measure. If only $\{a\}$ has nonzero measure then $E((a, b))=0$ and so

$$\sigma(A|E([a, b))H) = \sigma(A|E(\{a\})H)\subset\{a\}$$

which proves (ii) above since no bounded operator can have empty spectrum. If two sets on the right of $(*)$ have nonzero measure then clearly (i) follows from an appropriate choice of c. Thus we may assume that exactly one interval on the right of $(*)$ has nonzero measure; denote that interval by $[c_1, d_1)$. Partition this interval in a manner analogous to $(*)$ except that every subinterval has length less than $\frac{1}{3}$. As reasoned above, either the lemma is proved at this step or else there is exactly one subinterval, say $[c_2, d_2)$, with nonzero measure. Either this process terminates and the lemma is proved or else it continues indefinitely. Assume the latter and let the intervals constructed be $\{[a\ b), [c_1, d_1), [c_2, d_2), \dots\}$. Thus the sequence $\{E([a, b))$ $E([c_1, d_1)), E([c_2, d_2)), \dots\}$ consists of only one nonzero constant. By the Monotone Convergence Theorem (applied weakly) that constant is the measure of

$$S = \bigcap_{j=1}^{\infty}[c_j, d_j).$$

By the construction of the subintervals S consists of only one point, say $S=\{e\}$. Thus

$$\sigma(A|E([a, b))H) = \sigma(A|E(\{e\})H)\subset\{e\}$$

and equality follows since the bounded operator cannot have empty spectrum.

4.3. Lemma. *Let $\{Q_1, Q_2, \ldots\}$ be a countably infinite set of mutually orthogonal nonzero projections onto subspaces of $A_0 H$ which reduce A_0. Then the set $\mathscr{C}=\{A_0 Q_k:\ k=1, 2, \ldots\}$ is linearly independent over the real numbers.*

Proof. Assume that $c_1, \ldots, c_m$ are constants such that $\sum_{j=1}^{m} c_j A_0 Q_{n(j)}=0$ or equivalently

$$0 = A_0 \sum_{j=1}^{\infty} c_j Q_{n(j)}.$$

Since $Q_{n(j)} H \subset A_0 H$ and A_0 is one-to-one on $A_0 H$, we see that $A_0 Q_{n(j)} \neq 0$. Furthermore, since the projections are mutually orthogonal, for each j we can choose a vector f such that $Q_{n(j)} f \neq 0$. Thus every c_j is zero and this proves the linear independence of $\mathscr{C}$.

4.4. Lemma. *If $A_0 H$ is infinite dimensional and the spectrum of A_0 is not a finite set then the convex set of all positive near-approximants of T is infinite dimensional.*

Proof. Let $\mathscr{B}$ consist of all collections of disjoint intervals, for example $\mathscr{S}= =\{I_\gamma:\ \gamma\in\Gamma\}$, where $I_\gamma=[a_\gamma, b_\gamma)$ and $E(I_\gamma)\neq 0$ for all $\gamma\in\Gamma$ and $E(\cdot)$ is the spectral measure of A_0. Then $\mathscr{B}$ is partially ordered by inclusion and Zorn's lemma easily shows the existence of a maximal element $\mathscr{S}_0$. For the sake of obtaining a contradiction assume that $\mathscr{S}_0$ is finite, say $\mathscr{S}_0=\{I_1, \ldots, I_m\}$ with $I_k=[a_k, b_k)$ for $k=1, \ldots, m$. If we could find $c_k\in(a_k, b_k)$ for some k such that

$$E([a_k, c_k)) \neq 0 \quad \text{and} \quad E([c_k, b_k)) \neq 0$$

then we would contradict the maximality of $\mathscr{S}_0$. Thus we may appeal to Lemma 4.2 and conclude that there exists $e_k\in I_k$ for $k=1, \ldots, m$ such that

$$\sigma\big(A_0|E(J)H\big)\subset\{e_1, \ldots, e_m\}$$

where $J=\bigcup_{k=0}^{m} I_k$. Since

$$\sigma(A_0) = \sigma\big(A_0|E(J)H\big)\cup\sigma\big(A_0|E(J')H\big)$$

with $J'=[0, 2\,\|A_0\|)\setminus J$ and since the spectrum of A_0 is not a finite set, it follows that $\sigma\big(A_0|E(J')H\big)$ is nonempty and $E(J')$ is nonzero. From the form of the intervals I_k and the definitions of J and of J' it is clear that we can find an interval $[\alpha, \beta)$ contained in J' with the property that $E([\alpha, \beta))\neq 0$. This contradicts the maximality of $\mathscr{S}_0$ and consequently $\mathscr{S}_0$ must be infinite. Let $\mathscr{S}_1$ be a countably infinite collection of intervals belonging to $\mathscr{S}_0$ with the property that zero does not belong to any interval, say $\mathscr{S}_1=\{I_1, I_2, \ldots\}$.

Let $\mathscr{C}=\{A_0 E(I_n):\ n=1, 2, \ldots\}$. Because $(\delta^2-C^2)^{1/2}$ commutes with A_0 it commutes with $E(I_n)$ for $n=1, 2, \ldots$ and consequently Lemma 4.1 shows that each element of $\{P_0-C: C\in\mathscr{C}\}$ is a positive near-approximant. Since zero does not belong to I_n we have $E(I_n)H\subset A_0H$ and Lemma 4.3 shows that $\mathscr{C}$ is linearly independent over the reals. If $\mathscr{H}+S$ is any translate of a vector space $\mathscr{H}$ such that $\mathscr{H}+S\supset\mathscr{P}'(T)$ then 0 belongs to $\mathscr{H}+S-P_0$. Thus $S-P_0$ belongs to $\mathscr{H}$ thus $\mathscr{H}+S-P_0$ is actually $\mathscr{H}$ and so $\mathscr{H}\supset -\mathscr{C}$ consequently

$$\dim\mathscr{H}\geqq\dim\mathscr{C}.$$

Hence $\mathscr{P}'(T)$ is an infinite dimensional convex set.

4.5. Lemma. *Let $\mathscr{P}'(T)$ denote the set of positive near-approximants of $T=B+iC$ with $B=B^*$, $C=C^*$. If the dimension of H_0 is a finite positive integer p then the dimension of the convex set $\mathscr{P}'(T)$ is not greater than p^2.*

Proof. An operator on H_0 is self adjoint if its matrix (with respect to any basis) is conjugate symmetric; this fact can be used to give a basis for the self adjoint operators on H_0 considered as a real vector space. The number of elements in that basis is p^2, and consequently by the argument in the last two sentences of the proof of Lemma 4.4 it would suffice to prove that

$$(P_0-P)H\subset H_0$$

for every $P\in\mathscr{P}'(T)$. According to Corollary 3.2 of [3] we know that $P_0\geqq P_0-P\geqq 0$ and so $\ker P_0\subset\ker(P_0-P)$, consequently $\big((P_0-P)H\big)^-\subset(P_0H)^-$ for every $P\in\mathscr{P}'(T)$ Clearly it would suffice to prove that $\big((P_0-P)H\big)^-$ is contained in $\big((\delta^2-C^2)^{1/2}H\big)^-$. Because

$$\delta^2=|||T-P|||^2=\|(B-P)^2+C^2\|$$

thus $\delta^2-C^2\geqq(B-P)^2$, and so we have

$$(\delta^2-C^2)^{1/2}\geqq|B-P|\geqq B-P,$$

consequently

$$2(\delta^2-C^2)^{1/2}\geqq P_0-P.$$

By the above argument the lemma is proved.

4.6. Lemma. *The convex set $\mathscr{P}'(T)$ is infinite dimensional over the reals if and only if H_0 is an infinite dimensional subspace.*

Proof. Recall that the operator A_0 constructed by appeal to Lemma 3.2 has the property that

$$(*)\qquad (A_0H)^-=H_0$$

and so the hypothesis implies that A_0H is infinite dimensional. The case that the spectrum of A_0 is not a finite set is handled by Lemma 4.4 and consequently we may assume that

$$\sigma(A_0)=\{\lambda_1,\ldots,\lambda_p\}.$$

Since A_0 is self adjoint it is easily seen that each λ_j is an eigenvalue and

$$A_0 H = (A_0 H)^- = E(\{\lambda'_1, \ldots, \lambda'_e\})H$$

where $\{\lambda'_1, \ldots, \lambda'_e\}$ is the set of nonzero eigenvalues of A_0 and $E(\cdot)$ is the spectral measure of A_0. Thus one of the nonzero eigenvalues, say λ'_i, has infinite multiplicity.

Because P_0 and $(\delta^2 - C^2)^{1/2}$ commute with A_0 they commute with $E(\{\lambda'_1\})$; $E(\{\lambda'_1\})H$, which we shall denote by H_1, reduces P_0 and $(\delta^2 - C^2)^{1/2}$. In view of $(*)$ we may assume that $(\delta^2 - C^2)^{1/2} H_1$ is infinite dimensional. Thus if the spectrum of $(\delta^2 - C^2)^{1/2}|H_1$ is finite then it has a nonzero eigenvalue with infinite multiplicity. Clearly we can find a countably infinite collection of mutually orthogonal projections onto subspaces of H_1 which reduce $(\delta^2 - C^2)^{1/2}|H_1$, say $Q_1, Q_2, \ldots$. Since $Q_k H$ reduces A_0 and is contained in $A_0 H_1$ for $k = 1, 2, \ldots$, we see by Lemma 4.3 that $\mathscr{C} = \{A_0 Q_k : k = 1, 2, \ldots\}$ is a linearly independent set over the reals. Because Q_k commutes with $(\delta^2 - C^2)^{1/2}$ we conclude from Lemma 4.1 each element of $\{P_0 - C : C \in \mathscr{C}\}$ is a positive near-approximant. Thus in the case that $(\delta^2 - C^2)^{1/2}|H_1$ has finite spectrum $\mathscr{C}$ is an infinite set of linearly independent positive operators.

If $(\delta^2 - C^2)^{1/2}|H_1$ does not have finite spectrum then we can exploit Lemma 4.2 as was done in the first paragraph of the proof of Lemma 4.4 to obtain a countably infinite set of mutually orthogonal projections onto subspaces of H_1 which reduce $(\delta^2 - C^2)^{1/2}$. As in the paragraph above we deduce the existence of an infinite set of positive operators which is linearly independent over the reals. By the argument in the last two sentences of the proof of Lemma 4.4, we have shown $\mathscr{P}'(T)$ to be an infinite dimensional convex set provided that H_0 is an infinite dimensional subspace.

If H_0 were a nontrivial finite dimensional subspace then we could conclude from Lemma 4.5 that $\mathscr{P}'(T)$ is a finite dimensional convex set. If H_0 were trivial then Theorem 4.2 in [3] implies that $\mathscr{P}'(T) = \{P_0\}$. Hence this lemma is proved.

4.7. Lemma. *The subspace H_0 is finite dimensional then it reduces the operator $\left(\frac{1}{2} A_0 - (\delta^2 - C^2)^{1/2}\right)^2 + C^2$; if in addition S denotes the restriction of that operator to H_0 then $\|S\| < \delta^2$.*

Proof. Since $(A_0 H)^- = H_0$ and H_0 is finite dimensional, we have $A_0 H = H_0$. Because A_0 commutes with C^2 we see that H_0 is invariant under C^2 and $(\delta^2 - C^2)^{1/2}$; thus H_0 reduces $\left(\frac{1}{2} A_0 - (\delta^2 - C^2)^{1/2}\right)^2 + C^2$.

Note that Lemma 4.1 implies that $P_0 - \frac{1}{2} A_0$ is a positive near-approximant for T and consequently we know that

$$\|S\| = \left\| \left(T - P_0 + \frac{1}{2} A_0 \right) \Big| H_0 \right\|^2 \leqq \delta^2.$$

If equality holds in the above inequality then δ^2 is an eigenvalue of S. In order to obtain a contradiction we assume this. Let $H_1=\ker(S-\delta^2)$ and take $0\neq h\in H_1$. By the commutativity of A_0 and C^2 we have

$$\left(\frac{1}{4}A_0^2+A_0(\delta^2-C^2)^{1/2}+(\delta^2-C^2)\right)h=\left(\frac{1}{2}A_0-(\delta^2-C^2)^{1/2}\right)^2h=(\delta^2-C^2)h$$

or $\left(\frac{1}{4}A_0-(\delta^2-C^2)^{1/2}\right)A_0h=0$. Since $\frac{1}{4}A_0-(\delta^2-C^2)^{1/2}$ is obviously invertible, it must be that $h\in\ker A_0$. On H_0 the operator A_0 is one-to-one and so $h=0$. This contradiction proves the lemma.

4.8. Lemma. *If the dimension of H_0 is the finite positive integer p then*

$$\dim \mathscr{P}'(T)=p^2.$$

Proof. It was established in Lemma 4.5 that p^2 is an upper bound for the dimension of $\mathscr{P}'(T)$. Recall that A_0 commutes with both P_0 and $(\delta^2-C^2)^{1/2}$; also we have $(A_0H)^-=H_0$. By the finite dimensionality we have $A_0H=H_0$ and consequently H_0 is invariant under P_0 and $(\delta^2-C^2)^{1/2}$; thus H_0 reduces A_0 and $(\delta^2-C^2)^{1/2}$ and we can simultaneously diagonalize $A_0|H_0$ and $(\delta^2-C^2)^{1/2}|H_0$. Let $\{e_1, \ldots, e_p\}$ be an orthonormal basis which simultaneously diagonalizes the two restrictions above and let Q_k be the orthogonal projection onto e_k for $k=1, \ldots, p$. Note that each $Q_k|H_0$ commutes with both $A_0|H_0$ and $(\delta^2-C^2)^{1/2}|H_0$ and $Q_kH\subset A_0H$. By Lemma 4.1 $\{P_0-A_0Q_k\colon k=1, \ldots, p\}$ consists of positive near-approximants and the argument used to prove Lemma 4.3 shows that $\{A_0Q_k\colon k=1, \ldots, p\}$ is a linearly independent set.

Since H_0 reduces $(\delta^2-C^2)^{1/2}$ it clearly reduces C^2 and we may apply Lemma 4.7. Assume that A_0 restricted to H_0 is diag $\{2a_1, \ldots, 2a_p\}$ with $a_1\geqq a_2\geqq\ldots\geqq a_p$ and note that a_p is positive since $H_0=A_0H$. For any positive γ not greater than a_p define A_γ on H_0 by

$$A_\gamma=\operatorname{diag}\{a_1-\gamma, \ldots, a_p-\gamma\}+\langle\cdot, e_k\rangle\gamma e_j+\langle\cdot, e_j\rangle\gamma e_k$$

for any pair of $j, k=1, \ldots, p$ and $k>j$. Since A_γ is obviously self adjoint and converges to $\frac{1}{2}A_0$ in operator norm, the upper semicontinuity of the spectrum of $\frac{1}{2}A_0$ shows that A_γ is nonnegative for all γ sufficiently small. Because the norm of

$$\langle\cdot, e_k\rangle e_j+\langle\cdot, e_j\rangle e_k$$

is one, it is easy to see that $\frac{1}{2}A_0-A_\gamma$ is nonnegative. The continuity of the expression

$$(X-(\delta^2-C^2)^{1/2})^2+C^2$$

in X with respect to the operator norm and Lemma 4.7 show that

$$\|(A_\gamma-(\delta^2-C^2)^{1/2})^2+C^2\| \leqq \delta^2$$

for all γ sufficiently small. It now follows that for all γ sufficiently small we have

$$0 \leqq A_\gamma \leqq \frac{1}{2} A_0 \leqq P_0$$

$$|||T-(P_0-A_\gamma)|||^2 = \|(A_\gamma-(\delta^2-C^2)^{1/2})^2+C^2\| \leqq \delta^2$$

where A_γ has been extended to all of H by making it zero on the orthogonal complement of H_0. Thus P_0-A_γ is a positive near-approximant of T. An analogous argument shows that $P_0-A'_\gamma$ is a positive near-approximant when A'_γ is zero on $(H_0)^\perp$ and its restriction to H_0 is

$$\operatorname{diag}\{a_1-\gamma, \dots, a_p-\gamma\} + \langle \cdot, e_k\rangle i\gamma e_j - \langle \cdot, e_j\rangle i\gamma e_k$$

and γ is sufficiently small. The linear independence over the reals of the following set is apparent:

$$\{A_0 Q_i, A_\gamma, A'_\gamma\colon i = 1, \dots, p \text{ and } k > j\}.$$

There are p^2 operators in this set and the argument in the last two sentences of the proof of Lemma 4.4 shows that the dimension of $\mathscr{P}'(T)$ as a convex set is at least p^2. Equality then follows from Lemma 4.5.

Proof of Theorem 3.1. This theorem follows from Lemma 4.6 if p is infinite; it follows from Lemma 4.8 if p is a finite positive integer; it follows from Theorem 4.2 of [3] if p equals zero.

§ 5. Consequences of § 4 for positive approximants

The method used in the preceding section to construct positive near-approximants was initiated as a method for constructing positive approximants in Theorem 4.3 of [2]. The object of this section is to show that the construction of near-approximants in the preceding section can be refined so that approximants result. First we must prove a result analogous to Lemma 4.7 which can be applied to the operator norm just as Lemma 4.7 was applied to the new norm.

5.1. Lemma. *If A_0 commutes with C and H_0 is finite dimensional then H_0 reduces $\frac{1}{2} A_0-(\delta^2-C^2)^{1/2}+iC$ and if S denotes the restriction of that operator to H_0 then we have $\|S\|<\delta$.*

Proof. Since $(A_0H)^- = H_0$ and H_0 is finite dimensional, we have $A_0H = H_0$. Because A_0 commutes with C we know that H_0 reduces C, $(\delta^2 - C^2)^{1/2}$ and $\frac{1}{2}A_0 - -(\delta^2 - C^2)^{1/2} + iC$. Moreover, the last operator is normal and consequently S is normal. Because the numerical radius of S equals $\|S\|$, Theorem 3.1 of [3] implies that $|||S|||$ equals $\|S\|$. The desired conclusion now follows from Lemma 4.7.

Now we give our basic theorem for positive approximants.

5.2. Theorem. *Let T be a normal operator and let p be the dimension of the subspace H_0. If $\mathscr{P}(T)$ denotes the convex set of positive approximants of T then its real dimension is p^2. Here all infinite cardinal numbers are identified.*

Proof. By Lemma 4.5 it is immediate that the dimension of $\mathscr{P}(T)$ is not greater than p^2; recall that $\mathscr{P}(T)$ is contained in $\mathscr{P}'(T)$. The equality will be established when we show that each positive near-approximant previously constructed is in fact a positive approximant. Note that each positive near-approximant constructed in the proofs of Lemma 4.4, Lemma 4.6 and the first part of the proof of Lemma 4.8 has form $P_0 - A_0Q$ where Q is an orthogonal projection commuting with A_0 and $(\delta^2 - C^2)^{1/2}$. Because T is normal, B and C commute. Furthermore, it is straightforward to see that Lemma 3.2 implies that there is a positive operator A_1 dominated by $\sqrt{2}(\delta - C)^{1/2}$, $\sqrt{2}(\delta + C)^{1/2}$ and $P_0^{1/2}$; clearly A_1 commutes with C and with P_0. If we set $A_0 = (A_1)^2$ then it is routine to see that this A_0 has all the properties of the previous A_0 and also it commutes with C and B. It follows that $\{A_0, B, C, P_0\}$ is a set of commuting operators and consequently each commutes with all spectral projections of the others. If we take Q to be a spectral projection for one of the above operators then

$$\|T - (P_0 - A_0Q)\|^2 = \|-(\delta^2 - C^2)^{1/2} + iC + A_0Q\|^2 =$$
$$= |||-(\delta^2 - C^2)^{1/2} + iC + A_0Q|||^2 \leqq \delta^2$$

since both norms agree on normal operators. (Recall Theorem 3.1 of [3].) An examination of those previous constructions shows that Q can be taken to be such a spectral projection of either A_0 or $(\delta^2 - C^2)^{1/2}$ except possibly when $(\delta^2 - C^2)^{1/2}$ restricted to an infinite dimensional eigenspace of A_0 has an infinite dimensional eigenvalue. In the latter case restrict P_0 to the infinite dimensional eigenspace of $(\delta^2 - C^2)^{1/2}$ and use either the spectral projections or else projections onto arbitrary eigenvectors of P_0. Thus if H_0 is infinite dimensional it is established that $\mathscr{P}(T)$ is infinite dimensional. Of course, if H_0 is $\{0\}$ then $\mathscr{P}'(T)$ is just $\{P_0\}$ by Theorem 4.2 of [3] and necessarily $\mathscr{P}(T) = \{P_0\}$ which proves this theorem in that case.

The only remaining case requires that p be a finite positive integer. The first positive near-approximants constructed in Lemma 4.8 have the form $P_0 - A_0Q$ and we may use the argument above to guarantee that each $P_0 - A_0Q$ is actually a positive approximant. It would suffice to show that each P_0-A_γ and $P_0-A'_\gamma$ constructed in the proof of Lemma 4.8 is a positive approximant. The arguments given in that earlier proof show that

$$0 \leqq A_\gamma \leqq A_0 \leqq P_0 \quad \text{and} \quad 0 \leqq A'_\gamma \leqq A_0 \leqq P_0$$

for all positive γ sufficiently small. The continuity of the expression

$$X-(\delta^2-C^2)^{1/2}+iC$$

in X with respect to the operator norm and Lemma 5.1 show that for γ sufficiently small

$$\|A_\gamma-(\delta^2-C^2)^{1/2}+iC\| \leqq \delta \quad \text{and} \quad \|A'_\gamma-(\delta^2-C^2)^{1/2}+iC\| \leqq \delta.$$

Thus each P_0-A_γ and each $P_0-A'_\gamma$ is a positive approximant. The real linear independence of the set

$$\{A_0Q_i, A_\gamma, A'_\gamma: i = 1, \ldots, p \text{ and } k > j\}$$

proves the theorem.

It is immediate from the preceding theorem and Theorem 3.1 that $\mathscr{P}(T)$ and $\mathscr{P}'(T)$ have the same dimension when T is normal. Although it is apparent that $\mathscr{P}(T)$ is contained in $\mathscr{P}'(T)$, we are unable to determine when the two sets must coincide in general. From Theorem 5.2 of [3] the two convex sets coincide if T has a unique positive approximant or if T has a unique positive near-approximant. In the sixth section of this paper we shall show that the two convex sets coincide if either is one dimensional. The difficulty of handling the general case centers around the positive near-approximants which do not commute with A_0.

§ 6. Extreme Points of $\mathscr{P}(T)$

Before we give our main result on extreme points we state the following lemma which is a consequence of well-known results.

6.1. Lemma. *Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis which simultaneously diagonalizes the commuting positive operators R_1, R_2, R_3 on the Hilbert space H_0. Then the lower bound operator for R_1, R_2, R_3 constructed by Lemma 3.2 is*

$$\operatorname{diag}\{\mu_1, \ldots, \mu_n\}$$

relative to $\{e_1, \ldots, e_n\}$ where μ_j is $\min\{\langle R_ie_j, e_j\rangle: i=1, 2, 3\}$ *for $j=1, \ldots, n$.*

6.2. Theorem. *Assume that T is normal and that H_0 is finite dimensional. If $\{e_1, \ldots, e_p\}$ is an orthonormal basis which diagonalizes the restrictions of A_0, P_0 and C to H_0 and if Q is the orthogonal projection onto e_k then P_0-A_0Q is an extreme point of each of the sets $\mathscr{P}(T)$ and $\mathscr{P}'(T)$.*

Proof. It was indicated in the first paragraph of the proof of Theorem 5.2 that P_0-A_0Q is a positive near-approximant of T because Q commutes with A_0 and $2(\delta^2-C^2)^{1/2}$. As was shown in that proof the commutativity of Q and C implies that P_0-A_0Q is a positive approximant of T. Let $\mathscr{C}$ denote the convex set $\{P_0-P: P\in\mathscr{P}'(T)\}$. Clearly it suffices to show that A_0Q is an extreme point of $\mathscr{C}$.

Take $P\in\mathscr{P}'(T)$ and note that

$$\delta^2 = \|T-P\|^2 = \|(B-P)^2+C^2\|$$

and so $\delta^2\geqq(B-P)^2+C^2$. Since taking square roots is a monotone operator function we have

$$(\delta^2-C^2)^{1/2} \geqq \left((B-P)^2\right)^{1/2} = |B-P|$$

and it is easily seen that $|B-P|\geqq B-P$ and $|B-P|\geqq P-B$. Thus

$$(\delta^2-C^2)^{1/2} \geqq P-B, \quad P_0 = B+(\delta^2-C^2)^{1/2} \geqq P \geqq 0$$

and

$$(\delta^2-C^2)^{1/2} \geqq B-P, \quad 2(\delta^2-C^2)^{1/2} \geqq P_0-P \geqq 0.$$

Thus P_0-P, which we shall denote by A, is dominated by both P_0 and $2(\delta^2-C^2)^{1/2}$. It follows that:

$$\ker A \supset \operatorname{span}\{\ker P_0, \ker(\delta^2-C^2)^{1/2}\}$$

or

$$(AH)^- \subset (P_0H)^- \cap \left((\delta^2-C^2)^{1/2}H\right)^- = H_0.$$

Apply Lemma 6.1 to the derivation of A_1 in the first paragraph of the proof of Theorem 5.2. It follows that every eigenvalue of A_1 is an eigenvalue of one of the operators $\sqrt{2}(\delta-C)^{1/2}$, $\sqrt{2}(\delta+C)^{1/2}$ and $P_0^{1/2}$ with common eigenvectors and so each eigenvalue of A_0 is an eigenvalue of one of the two operators P_0 and $2(\delta^2-C^2)^{1/2}$ with the same eigenvectors. From the preceding paragraph we know that $P\in\mathscr{P}'(T)$ implies that

(I) $$AH\subset(AH)^- = \left((P_0-P)H\right)^- \subset H_0$$

and $0\leqq A\leqq P_0$, $A\leqq 2(\delta^2-C^2)^{1/2}$.

Now assume that $A_0Q=\lambda A_2+(1-\lambda)A_3$ with $\lambda\in(0, 1)$ and $A_2, A_3\in\mathscr{C}$. By the preceding paragraph we have

(II) $$\langle A_2e_k, e_k\rangle \leqq \langle A_0e_k, e_k\rangle = \langle A_0Qe_k, e_k\rangle, \quad \langle A_3e_k, e_k\rangle \leqq \langle A_0Qe_k, e_k\rangle.$$

For $e_j\neq e_k$ we have

$$0 \leqq \lambda\langle A_2e_j, e_j\rangle+(1-\lambda)\langle A_3e_j, e_j\rangle = \langle(\lambda A_2+(1-\lambda)A_2)e_j, e_j\rangle = \langle A_0Qe_j, e_j\rangle = 0$$

and so $\langle A_2e_j, e_j\rangle=0=\langle A_3e_j, e_j\rangle$ for $e_j\neq e_k$.

From this and (I) it follows that

$$A_j = \langle\cdot, e_k\rangle\mu_{jk}e_k \text{ for } j = 2, 3.$$

From (II) we conclude that

(III) $$\mu_{jk} \leqq \langle A_0 Q e_k, e_k \rangle \text{ for } j = 2, 3$$

and if either inequality were strict it would certainly follow that

$$\lambda A_2 + (1-\lambda) A_3 \neq A_0 Q.$$

Hence equality holds in each inequality of (III); it follows that $A_2 = A_0 Q = A_3$. Apparently, $A_0 Q$ is an extreme point of $\mathscr{C}$ and the theorem is proved.

As we noted earlier the preceding theorem gives a characterization of those normal operators T for which the dimension of $\mathscr{P}(T)$ is one. In that circumstance it also gives a very explicit description of both $\mathscr{P}(T)$ and $\mathscr{P}'(T)$.

6.3. Corollary. *Assume that T is normal and that H_0 is a one dimensional subspace. Let f_0 be a unit vector in H_0 and let λ_0 and A_1 be defined by the equations*

$$\lambda_0 = \min \{\langle P_0 f_0, f_0 \rangle, \langle 2(\delta^2 - C^2)^{1/2} f_0, f_0 \rangle\},$$
$$A_1 = \langle \cdot, f_0 \rangle \lambda_0 f_0.$$

Then $\mathscr{P}(T)$ and $\mathscr{P}'(T)$ conicide with the convex hull of P_0 and $P_0 - A_1$; consequently we have

$$\mathscr{P}'(T) = \mathscr{P}(T) = \{P_0 - \lambda A_1 \colon \lambda \in [0, 1]\}.$$

Proof. In the second paragraph of the proof of the preceding theorem it was shown that P_0 is an absolutely maximal element of $\mathscr{P}'(T)$ — that is $P \in \mathscr{P}'(T)$ implies $P \leqq P_0$. Obviously P_0 has the same property for $\mathscr{P}(T)$ and it easily follows that P_0 is an extreme point of both sets. In view of the integral third paragraph given in the proof of Theorem 6.2 we see that A_1 above is actually the operator A_0 in Theorem 6.2. By that theorem it follows that $P_0 - A_1$ is an extreme point of both of the sets $\mathscr{P}'(T)$ and $\mathscr{P}(T)$. In view of Theorem 3.1 and Theorem 5.2 the real dimension of each of the convex sets $\mathscr{P}'(T)$ and $\mathscr{P}(T)$ is one and geometrically it is clear that both sets must be the convex hull of P_0 and $P_0 - A_1$. For completeness sake we prove this last assertion. The set $\mathscr{C} = \{P_0 - P \colon P \in \mathscr{P}'(T)\}$ is one dimensional and contains the zero operation; thus it is spanned by any nonzero operator in the set, for example A_1. So $\mathscr{C} \subset \{cA_1 \colon c \text{ real}\}$. However, $A \in \mathscr{C}$ implies that

$$0 \leqq A \leqq P_0, \quad 0 \leqq A \leqq 2(\delta^2 - C^2)^{1/2}$$

by the argument given in the second paragraph of the proof of Theorem 6.2. It follows that

$$\mathscr{C} \subset \{cA_1 \colon c \in [0, 1]\}$$

by the construction of A_1. Hence $\mathscr{P}'(T)$ is contained in the set

$$\{(1-\lambda) P_0 + \lambda (P_0 - A_1) = P_0 - \lambda A_1 \colon \lambda \in [0, 1]\}.$$

Since $\mathscr{P}'(T)$ is convex and both P_0 and $P_0 - A_1$ are positive approximants, it must be that both $\mathscr{P}'(T)$ and $\mathscr{P}(T)$ coincide with the above set.

§ 7. Open questions

For T a normal operator and H_0 a finite dimensional subspace, where H_0 was defined in section three, we constructed a real basis for the convex set $\mathscr{P}'(T)$ and each element of that basis is a positive approximant. This tends to suggest that $\mathscr{P}(T)$ and $\mathscr{P}'(T)$ might coincide. We now show that $\mathscr{P}'(T)$ can properly contain $\mathscr{P}(T)$. Let T be the four dimensional operator defined by the diagonal matrix diag $\{i, -i, 2i, -2i\}$. In this instance $T=iC$ and $P_0=\text{diag}\,\{\sqrt{3}, \sqrt{3}, 0, 0\}$.

Let A be the 4×4 matrix (a_{ij}) with $a_{11}=a_{12}=a_{21}=a_{22}=\sqrt{3}/2$ and all other entries equal to zero. Then P_0-A is a positive near-approximant for T but it is not a positive approximant. Thus $\mathscr{P}(T)$ is properly contained in $\mathscr{P}'(T)$ and this gives rise to our first question.

Question 1. What characterizes those normal operators T for wich $\mathscr{P}(T)=$ $=\mathscr{P}'(T)$?

The procedure for obtaining basis elements of $\mathscr{P}'(T)$ which do not necessarily commute with A_0 is less explicit than the construction of basis elements which do commute with A_0. That observation and the remarks of the preceding paragraph suggest several questions.

Question 2. Assuming that H_0 is finite dimensional, what conditions on T suffice for $\mathscr{P}(T)$ to have only a finite number of extreme points? What suffices for $\mathscr{P}'(T)$ to have only a finite number of extreme points?

Our last question would be considerably more interesting if the preceding question had been answered.

Question 3. What is an extreme point of $\mathscr{P}'(T)$ which fails to commute with A_0?

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UNIVERSITY OF GEORGIA
ATHENS, GEORGIA 30602

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