An identity for Laguerre polynomials

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Dedicated to my loved father, Professor László Rédei, on the occasion of his seventy-fifth birthday.

We shall prove the following representation for Laguerre polynomials:

$$L_n(x) = (-1)^n \frac{e^x}{n!} \left(x \frac{d^2}{dx^2} + \frac{d}{dx}\right)^n e^{-x}. \tag{1}$$

(We use the same convention for $L_n(x)$ as in reference [1].) This representation for $L_n(x)$ is an analogue of the well known representation for Hermite polynomials:

$$H_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2}.$$

In spite of its simple and potentially useful form, we have not been able to find formula (1) in any of the standard texts.

Proof. Using the standard representation

$$L_n(x) = \frac{1}{n!} e^x \left(\frac{d}{dx}\right)^n (x^n e^{-x}) \tag{2}$$

we can put equation (1) into the equivalent form

$$A_n(x) = (-1)^n \left(\frac{d}{dx}\right)^n (x^n e^{-x}), \tag{3}$$

$A_n(x)$ being defined by

$$A_n(x) = \left(x \frac{d^2}{dx^2} + \frac{d}{dx}\right)^n e^{-x}.$$

We proceed by induction. Equation (3) is obviously true for $n=0$ and for $n=1$. We assume it to be true for n. It then follows that

$$A_{n+1}(x) = \left(x\frac{d^2}{dx^2}+\frac{d}{dx}\right)\left[(-1)^n\left(\frac{d}{dx}\right)^n(x^n e^{-x})\right] =$$

$$= (-1)^n\left(x\frac{d^{n+2}}{dx^{n+2}}(x^n e^{-x})+\frac{d^{n+1}}{dx^{n+1}}(x^n e^{-x})\right).$$

We use, for the first term in the right hand side of this equation, the identity

$$\left(\frac{d}{dx}\right)^n(xf(x)) = n\frac{d^{n-1}}{dx^{n-1}}f(x)+x\frac{d^n}{dx^n}f(x)$$

valid for any smooth function $f(x)$, to obtain that

$$A_{n+1}(x) = (-1)^n\left[\frac{d^{n+2}}{dx^{n+2}}(x^{n+1}e^{-x})-(n+1)\frac{d^{n+1}}{dx^{n+1}}(x^n e^{-x})\right].$$

Since

$$\frac{d}{dx}(x^{n+1}e^{-x}) = (n+1)x^n e^{-x}-x^{n+1}e^{-x}$$

it follows that

$$A_{n+1}(x) = (-1)^{n+1}\left(\frac{d}{dx}\right)^{n+1}(x^{n+1}e^{-x}). \qquad \text{Q.E.D.}$$

Reference

[1] Bateman Manuscript Project, *Higher Transcendental Functions*, vol. II (New York, 1953).

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