On the strong approximation of orthogonal series

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Dedicated to Professor Károly Tandori on his 50th birthday

Introduction

Let $\{\varphi_n(x)\}$ be an orthogonal system on the interval (a, b). We consider the orthogonal series

(1) $$\sum_{n=0}^{\infty} c_n \varphi_n(x) \quad \text{with} \quad \sum_{n=0}^{\infty} c_n^2 < \infty.$$

It is well known that the series (1) converges in L^2 to a square-integrable function $f(x)$. Let us denote the partial sums and the (C, α)-means of the series (1) by $s_n(x)$ and $\sigma_n^\alpha(x)$, respectively.

In [2] we proved that *if*

(2) $$\sum_{n=1}^{\infty} c_n^2 n^{2\gamma} < \infty \quad \textit{and} \quad 0 < \gamma < 1,$$

then

$$f(x) - \sigma_n^1(x) = o_x(n^{-\gamma})$$

almost everywhere in (a, b).

G. SUNOUCHI [4] generalized this result proving that *if* (2) *is satisfied, then*

(3) $$\left\{\frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |f(x) - s_\nu(x)|^k\right\}^{1/k} = o_x(n^{-\gamma})$$

holds almost everywhere in (a, b) *for any* $\alpha > 0$ *and* $0 < k < \gamma^{-1}$, *where* $A_n^\alpha = \binom{n+\alpha}{n}$.

This result was generalized in [3] in such a way that *we replaced the partial sums in* (3) *by* (C, δ)*-means, where* δ *can also be negative.* (See Theorem 1 of [3].)

In [3] (Theorem 2) we also proved that *if* $\sum_{n=1}^{\infty} c_n^2 n^{2\alpha} < \infty$ *with any positive* γ, *then*

(4) $$\left\{\frac{1}{n} \sum_{\nu=n}^{2n} |s_\nu(x) - f(x)|^k\right\}^{1/k} = o_x(n^{-\gamma})$$

holds almost everywhere in (a, b) *for any* $0 < k \leqq 2$.

The aim of the present paper is to generalize further these results.

We consider a regular summation method T_n determined by a triangular matrix $\|\alpha_{nk}/A_n\|$ $\left(\alpha_{nk}\geqq 0 \text{ and } A_n=\sum_{k=0}^{n}\alpha_{nk}\right)$, i.e. if s_k tends to s, then

$$T_n=\frac{1}{A_n}\sum_{k=0}^{n}\alpha_{nk}s_k\to s.$$

Theorem I. *Suppose that* $0<\gamma<1$ *and* $0<k<\gamma^{-1}$,

(5) $$\sum_{n=1}^{\infty}c_n^2 n^{2\gamma}<\infty,$$

furthermore that there exists a number $p>1$ *such that*

(6) $$\frac{p}{p-1}k\geqq 2$$

and with this p for any $0<\delta<1$ *and* $2^m<n\leqq 2^{m+1}$

(7) $$\sum_{l=0}^{m}\left\{\sum_{\nu=2^l-1}^{\min(2^{l+1},n)}\alpha_{n\nu}^p(\nu+1)^{p(1-\delta)-1}\right\}^{1/p}\leqq K\left(\sum_{\nu=0}^{n}\alpha_{n\nu}\right)n^{-\delta}.$$ [1]

Then for arbitrary

(8) $$\beta>1-\frac{p-1}{pk}$$

we have

(9) $$\left\{\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|f(x)-\sigma_\nu^{\beta-1}(x)|^k\right\}^{1/k}=o_x(n^{-\gamma})$$

almost everywhere in (a,b).

It is easy to verify that in the special case $\alpha_{n\nu}=A_{n-\nu}^{\alpha-1}$ $(\alpha>0)$ condition (7) is satisfied, thus with $\beta=1$ Theorem I contains the result of SUNOUCHI. It can be shown that Theorem I includes our result in connection with (C,δ)-means of negative order, too. Furthermore we have some corollaries:

Corollary 1. *Suppose that* $0<\gamma<1$, $0<k<\gamma^{-1}$, *and that* (5) *is satisfied. Then we have*

$$\left\{\frac{1}{n}\sum_{\nu=n}^{2n}|f(x)-\sigma_\nu^{\beta-1}(x)|^k\right\}^{1/k}=o_x(n^{-\gamma})$$

for any $\beta>1-\min(1/2,1/k)$ *almost everywhere in* (a,b).

Corollary 2. *Under the hypothesis of Theorem 1 we have*

$$\left\{\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|f(x)-\sigma_\nu^{\beta-1}(\{\mu_i\};x)|^k\right\}^{1/k}=o_x(n^{-\gamma})$$

[1]) $K, K_1, K_2, \ldots$ will denote positive constants not necessarily the same at each occurrence.

almost everywhere in (a, b) *for any* $\beta > 1-(p-1)/pk$ *and for any increasing sequence* $\{\mu_i\}$; *where*

$$\sigma_n^\alpha(\{\mu_i\}; x) = \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} s_{\mu_i}(x).$$

From Corollary 2 in the special case $\beta=1$ we obtain immediately

Corollary 3. *Under the conditions of Theorem 1 we have*

$$\left\{\frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |f(x)-s_{\mu_\nu}(x)|^k\right\}^{1/k} = o_x(n^{-\gamma}) \tag{10}$$

almost everywhere in (a, b) *for any increasing sequence* $\{\mu_\nu\}$.

In the special case $\alpha_{n\nu}=A_{n-\nu}^{\alpha-1}$ $(\alpha>0)$ Corollary 3 reduces to Theorem 3 of [3].

Under the restrictions $0<k\leqq 2$ and $\beta=1$, but for arbitrary positive γ, Corollary 1 can be generalized to very strong approximation. In fact we have

Theorem II. *Suppose that* $0<k\leqq 2$ *and* $\gamma>0$; *and that* (5) *holds. Then*

$$\left\{\frac{1}{n} \sum_{\nu=n}^{2n} |s_{\mu_\nu}(x)-f(x)|^k\right\}^{1/k} = o_x(n^{-\gamma}) \tag{11}$$

almost everywhere in (a, b) *for any increasing sequence* $\{\mu_\nu\}$.

It is clear that (11) is a generalized form of (4).

Finally we show that under certain restrictions on γ, and $\{c_n\}$ an estimate similar to (10) can be given with any not necessarily monotonic sequence $\{l_\nu\}$ of distinct non-negative integers. Namely we have

Theorem III. *Suppose that* $0<\gamma<1/2$, $0<k\leqq 2$ *and*

$$\sum_{n=4}^{\infty} c_n^2 n^{2\gamma} (\log\log n)^2 < \infty, \tag{12}$$

furthermore that

$$\left\{\sum_{\nu=0}^{n} (\alpha_{n\nu})^{2/(2-k)}\right\}^{(2-k)/2} \leqq K\left(\sum_{\nu=0}^{n} \alpha_{n\nu}\right) n^{-k/2}. \tag{13}$$ [2]

Then we have

$$\left\{\frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |s_{l_\nu}(x)-f(x)|^k\right\}^{1/k} = o_x(n^{-\gamma}) \tag{14}$$

almost everywhere in (a, b) *for any (not necessarily monotonic) sequence* $\{l_\nu\}$ *of distinct non-negative integers.*

Theorem III gives immediately

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[2]) If $k=2$ then (13) means that $\max_{0\leqq\nu\leqq n} \alpha_{n\nu} \leqq K\left(\sum_{\nu=0}^{n} \alpha_{n\nu}\right) n^{-1}$.

Corollary 4. *If* $0<\gamma<1/2$, $0<k\leqq 2$ *and* $\alpha>k/2$, *furthermore* (12) *is satisfied, then*

$$\left\{\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}|s_{l_\nu}(x)-f(x)|^k\right\}^{1/k} = o_x(n^{-\gamma})$$

almost everywhere in (a, b) *for any (not necessarily monotonic) sequence* $\{l_\nu\}$ *of distinct non-negative integers.*

§ 1. Lemmas

We require the following lemmas.

Lemma 1 ([1], p. 359). *Let* $r\geqq l>1$, $\bar\gamma>0$, $\bar\alpha>\bar\gamma-1$ *and* $\bar\beta\geqq\bar\alpha+l^{-1}-r^{-1}$. *Then*

$$\left\{\sum_{n=0}^{\infty}(n+1)^{r\bar\gamma-1}|\tau_n^{\bar\beta}(x)|^r\right\}^{1/r} \leqq K\left\{\sum_{n=0}^{\infty}(n+1)^{l\bar\gamma-1}|\tau_n^{\bar\alpha}(x)|^l\right\}^{1/l},$$

where $\tau_n^\alpha(x) = \alpha(\sigma_n^{\alpha-1}(x)-\sigma_n^\alpha(x))$.

Lemma 2 ([4], Lemma 1). *If*

$$\sum_{n=1}^{\infty} c_n^2 n^{2\gamma} < \infty \quad \textit{with} \quad 0<\gamma<1,$$

then

$$\int_a^b\left\{\sum_{n=0}^{\infty}(n+1)^{2\gamma-1}|\sigma_n^{\alpha-1}(x)-\sigma_n^\alpha(x)|^2\right\}dx \leqq K\sum_{n=1}^{\infty}c_n^2n^{2\gamma}$$

for any $\alpha>1/2$.

Lemma 3 ([3], Theorem 4). *If* $0<\gamma\leqq 1/2$, $0<k\leqq 2$, $k\gamma<1$ *and*

$$\sum_{n=4}^{\infty} c_n^2 n^{2\gamma}(\log\log n)^2 < \infty,$$

then

$$\left\{\frac{1}{n}\sum_{\nu=0}^{n}|s_{l_\nu}(x)-f(x)|^k\right\}^{1/k} = o_x(n^{-\gamma}) \tag{1.1}$$

almost everywhere in (a, b) *for any (not necessarily monotonic) sequence* $\{l_\nu\}$ *of distinct non-negative integers.*

Lemma 4. *Under the conditions of Theorem I we have the inequality*

$$\int_a^b\left\{\sup_{0\leqq n<\infty}\left(\frac{n^{k\gamma}}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|\sigma_\nu^{\beta-1}(x)-\sigma_\nu^\beta(x)|^k\right)^{1/k}\right\}^2 dx \leqq K\sum_{n=1}^{\infty}c_n^2n^{2\gamma}. \tag{1.2}$$

Proof of Lemma 4. Set $q=p/(p-1)$, then

$$qk\geqq 2 \quad \text{and} \quad \beta>1-\frac{1}{qk}. \tag{1.3}$$

Applying Hölder's inequality, by (7) and $0<\gamma k<1$ we obtain that

$$\sum_{\nu=0}^{n}\alpha_{n\nu}|\tau_\nu^\beta(x)|^k \leqq \left\{\sum_{\nu=0}^{n}\alpha_{n\nu}^p(\nu+1)^{(p/q)-\gamma kp}\right\}^{1/p}\times$$

(1.4)
$$\times\left\{\sum_{\nu=0}^{n}(\nu+1)^{\gamma kq-1}|\tau_\nu^\beta(x)|^{qk}\right\}^{1/q}\leqq$$

$$\leqq K\left(\sum_{\nu=0}^{n}\alpha_{n\nu}\right)n^{-\gamma k}\left\{\sum_{\nu=0}^{n}(\nu+1)^{\gamma kq-1}|\tau_\nu^\beta(x)|^{qk}\right\}^{1/q}$$

By (1.3) we can choose α^* such that

(1.5)
$$\beta-\frac{1}{2}+\frac{1}{qk}>\alpha^*>\frac{1}{2}.$$

By (1.5), $0<\gamma<1$ and $qk\geqq 2$ the conditions of Lemma 1 are fulfilled with $r=qk$, $l=2$, $\bar\gamma=\gamma$, $\alpha=\alpha^*$ and $\bar\beta=\beta$. Using Lemma 1 we get

(1.6)
$$\left\{\sum_{\nu=0}^{\infty}(\nu+1)^{\gamma kq-1}|\tau_\nu^\beta(x)|^{qk}\right\}^{1/q}\leqq K_1\left\{\sum_{\nu=0}^{\infty}(\nu+1)^{2\gamma-1}|\tau_\nu^{\alpha^*}(x)|^2\right\}^{1/2}.$$

Thus by (1.4), (1.5), (1.6) and Lemma 2 we have

$$\int_a^b\left\{\sup_{1\leqq n<\infty}\left(\frac{n^{\gamma k}}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|\tau_\nu^\beta(x)|^k\right)^{1/k}\right\}^2dx\leqq K_2\int_a^b\left\{\sum_{\nu=0}^{\infty}(\nu+1)^{2\gamma-1}|\tau_\nu^{\alpha^*}(x)|^2\right\}dx\leqq$$

$$\leqq K_3\sum_{n=1}^{\infty}c_n^2n^{2\gamma}<\infty,$$

which gives statement (1.2).

§ 2. Proof of the theorems and corollaries

Proof of Theorem I. First we show that (7) implies

(2.1)
$$\sum_{\nu=0}^{n}\alpha_{n\nu}(\nu+1)^{-\delta}\leqq KA_nn^{-\delta}$$

for any $0<\delta<1$. Indeed,

$$\sum_{\nu=0}^{n}\alpha_{n\nu}(\nu+1)^{-\delta}\leqq\sum_{l=0}^{m}\sum_{\nu=2^l-1}^{\min(2^{l+1},n)}\alpha_{n\nu}(\nu+1)^{-\delta}\leqq$$

$$\leqq\sum_{l=0}^{m}\left\{\sum_{\nu=2^l-1}^{\min(2^{l+1},n)}\alpha_{n\nu}^p(\nu+1)^{-\delta p}\right\}^{1/p}\cdot 2^{l/q}\leqq KA_nn^{-\delta}.$$

By conditions (6) and (8) $\beta>1/2$, so we have (see e.g. inequality (3) with $k=1$)

$$\sigma_n^\beta(x)-f(x)=o_x(n^{-\gamma}).$$

Hence and from (2.1) it follows

$$\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|\sigma_\nu^\beta(x)-f(x)|^k = o_x(n^{-\gamma k}), \tag{2.2}$$

which implies

$$\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|\sigma_\nu^{\beta-1}(x)-f(x)|^k \leqq \frac{K}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|\sigma_\nu^{\beta-1}(x)-\sigma_\nu^\beta(x)|^k + o_x(n^{-\gamma k}). \tag{2.3}$$

Now for any fixed positive ε we choose N such that

$$\sum_{n=N}^{\infty} c_n^2 n^{2\gamma} < \varepsilon^3. \tag{2.4}$$

Let us define two new series

$$\sum_{n=1}^{\infty} a_n\varphi_n(x) \quad \text{with} \quad a_n = \begin{cases} c_n & \text{for} \quad n \leqq N, \\ 0 & \text{for} \quad n > N, \end{cases} \tag{2.5}$$

and

$$\sum_{n=1}^{\infty} b_n\varphi_n(x) \quad \text{with} \quad b_n = \begin{cases} 0 & \text{for} \quad n \leqq N, \\ c_n & \text{for} \quad n > N. \end{cases} \tag{2.6}$$

Denote $\sigma_n^\beta(a;x)$ and $\sigma_n^\beta(b;x)$, respectively, the n-th Cesàro-means of order β of the series (2.5) and (2.6).

It is clear that

$$\sigma_n^\beta(x) = \sigma_n^\beta(a;x)+\sigma_n^\beta(b;x).$$

Applying Lemma 4 with the series (2.5) and γ' satisfying the conditions $\gamma<\gamma'<1$ and $k\gamma'<1$, we obtain that

$$\frac{n^{\gamma k}}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|\sigma_\nu^{\beta-1}(a;x)-\sigma_\nu^\beta(a;x)|^k \to 0 \tag{2.7}$$

almost everywhere in (a, b).

On the other hand using Lemma 4 and (2.4) we obtain

$$\int_a^b\left\{\sup_{0\leqq n<\infty}\left(\frac{n^{k\gamma}}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|\sigma_\nu^{\beta-1}(b;x)-\sigma_\nu^\beta(b;x)|^k\right)^{1/k}\right\}^2 dx \leqq K\varepsilon^3.$$

Hence

$$\operatorname{meas}\left\{x \,\middle|\, \limsup\left(\frac{n^{k\gamma}}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|\sigma_\nu^{\beta-1}(b;x)-\sigma_\nu^\beta(b;x)|^k\right)^{1/k} > \varepsilon\right\} \leqq K\varepsilon.$$

This and (2.7) imply

$$\frac{n^{\gamma k}}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|\sigma_\nu^{\beta-1}(x)-\sigma_\nu^\beta(x)|^k \to 0$$

almost everywhere in (a, b).

Collecting our results we obtain statement (9).

Proof of Corollary 1. It is easy to verify that if

$$\alpha_{n\nu} = \begin{cases} 0 & \text{for} \quad \nu \leqq n/2, \\ 1 & \text{for} \quad \nu > n/2, \end{cases}$$

then (7) holds for arbitrary $p>1$. Thus, if $\beta>1-\min(1/2, 1/k)$, (6) and (8) can be satisfied with a suitably chosen p, and the statement of Corollary 1 follows from (9) immediately.

Proof of Corollary 2. We define

$$C_n = \left(\sum_{i=\mu_{n-1}+1}^{\mu_n} c_i^2\right)^{1/2}$$

and

$$\Phi_n(x) = \begin{cases} C_n^{-1} \sum\limits_{i=\mu_{n-1}+1}^{\mu_n} c_i \varphi_i(x) & \text{for} \quad C_n \neq 0, \\ (\mu_n - \mu_{n-1})^{-1/2} \sum\limits_{i=\mu_{n-1}+1}^{\mu_n} \varphi_i(x) & \text{for} \quad C_n = 0. \end{cases}$$

It is clear that the system $\{\Phi_n(x)\}$ is also an orthonormal one and

$$\sum_{n=1}^{\infty} C_n^2 n^{2\gamma} < \infty$$

obviously. Since

$$S_n(x) = \sum_{k=1}^{n} C_k \Phi_k(x) = s_{\mu_n}(x),$$

applying Theorem I to the series $\sum\limits_{n=1}^{\infty} C_n \Phi_n(x)$, we obtain the statement of Corollary 2.

Proof of Theorem II. Applying inequality (4) to the series $\sum\limits_{n=1}^{\infty} C_n \Phi_n(x)$ defined above, we get (11).

Proof of Theorem III. If $k=2$, then for any ν $(\leqq n)$

$$\frac{\alpha_{n\nu}}{A_n} \leqq \frac{K}{n}$$

whence, by (1.1), the estimate (13) follows obviously.

If $k<2$, then we can choose $p=2/k$. Using Hölder's inequality with this p and $q=2/(2-k)$ we obtain that

$$\sum_{\nu=0}^{n} \alpha_{n\nu} |s_{l_\nu}(x) - f(x)|^k \leqq \left\{\sum_{\nu=0}^{n} \alpha_{n\nu}^q\right\}^{1/q} \left\{\sum_{\nu=0}^{n} |s_{l_\nu}(x) - f(x)|^{kp}\right\}^{1/p}.$$

Hence, by (13) and (1.1),

$$\left\{\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|s_{l_\nu}(x)-f(x)|^k\right\}^{1/k} \leqq K\left\{\frac{1}{n}\sum_{\nu=0}^{n}|s_{l_\nu}(x)-f(x)|^2\right\}^{1/2} = o_x(n^{-\gamma})$$

which is the required estimate.

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