On the topological characterization of transitive Lie group actions

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The problem to characterize among the transitive actions of locally compact groups those which are effected by Lie groups has been solved by D. MONTGOMERY and L. ZIPPIN [6], pp. 236—244. According to their result if a σ-compact group G is an effective and transitive topological transformation group of a locally compact space X such that G/G_0 is compact where G_0 is the identity component and X is finite dimensional, then G is a Lie group provided that X is locally connected. Actually what this result yields is a characterization of the transitive Lie group actions among those of the finite dimensional locally compact ones, since the assumption that X is finite dimensional implies that G is finite dimensional as well. Accordingly the attempt at a general solution seems to be justified and with this respect the following theorem is proved below: *Let a σ-compact group G with G/G_0 compact be an effective and transitive topological transformation group of a locally compact space X. Then G is a Lie group if X is locally contractible.* In spite of the fact that in general local contractibility is a much more restrictive assumption then local connectedness this theorem is not materially weaker then the above mentioned result of Montgomery and Zippin since in case of finite dimensional coset spaces of locally compact groups these two assumptions are equivalent.

One prerequisite for the proof of the above theorem is a practicable description of the local structure of coset spaces of locally compact groups. Since all known treatments of this subject assume the finite dimensionality of the coset space a completely reshuffled approach has to be applied here. By a well-known result of H. YAMABE [9] a locally compact group always contains an open subgroup which can be approximated by Lie groups; thus local questions generally reduce to the case of such groups. Accordingly first a detailed study of the local structure of groups approximated by Lie groups is carried out below. Then on account of the results of this study the required description of the local structure of coset spaces of locally compact groups is obtained. Using this description the characterization of transitive Lie group actions is given at last.

1. The local structure of groups which can be approximated by Lie groups

According to the standard definition a topological group G is said to be *approximated by Lie groups* if a well-ordered index set Δ having a first element 1 and to any $\alpha \in \Delta$ a compact invariant subgroup A_α of G is given so that

$G_\alpha = G/A_\alpha$ is a Lie group
$A_\alpha \supset A_\beta$ for $\alpha, \beta \in \Delta$ with $\alpha < \beta$
$\{e\} = \bigcap \{A_\alpha | \alpha \in \Delta\}$ where $e \in G$ is the identity.

This terminology is based on the following well-known fact: Let $\pi_\alpha : G \to G_\alpha$ be the canonical epimorphism and $\pi_\alpha^\beta : G_\beta \to G_\alpha$ the epimorphism defined by $\pi_\alpha = \pi_\alpha^\beta \circ \pi_\beta$ for $\alpha, \beta \in \Delta$ with $\alpha < \beta$, then $\{G_\alpha, \pi_\alpha^\beta\}$ is an inverse system of Lie groups and its projective limit $G_\infty = \varprojlim \{G_\alpha | \alpha \in \Delta\}$ is isomorphic to G under the isomorphism $\omega : G \to G_\infty$ which is given by $\omega(g) = \{\pi_\alpha(g) | \alpha \in \Delta\}$ for $g \in G$.

Some subsequent arguments take advantage of the fact that an inverse system of Lie groups approximating a topological group can be adjusted in a certain sense. The precise description of this adjustment is given by the following

Lemma 1. *Let a system $\{A_\alpha | \alpha \in \Delta\}$ of compact invariant subgroups define an approximating inverse system of Lie groups for the topological group G and consider an index ε with $1 < \varepsilon \in \Delta$. Then there is a natural number $k > 1$ and to the well-ordered index set $E = \{1, \dots, k\} \cup \{\alpha | \alpha > \varepsilon, \alpha \in \Delta\}$ a system $\{A_\varrho^* | \sigma \in E\}$ of compact invariant subgroups defining an approximating inverse system of Lie groups for G and such that*

1. *$A_1^* = A_1$, $A_k^* = A_\varepsilon$ and $A_\sigma^* = A_\sigma$ for $\sigma \in E$ with $\sigma > k$,*
2. *A_1^*/A_2^* is finite, A_i^*/A_{i+1}^* for $i = 2, \dots, k-2$ is a 1-dimensional torus and A_{k-1}^*/A_k^* is a 1-dimensional torus or a compact connected semisimple Lie group.*

Proof. Consider the identity component $(A_1^\varepsilon)_0$ of the compact Lie group $A_1^\varepsilon = A_1/A_\varepsilon$ then $A^2 = \pi_\varepsilon^{-1}((A_1^\varepsilon)_0)$ is a compact invariant subgroup of G. Moreover A^2/A_ε is a compact connected Lie group, A_1/A^2 is finite and G/A^2 is a Lie group on account of the isomorphisms

$$A^2/A_\varepsilon \simeq (A_1^\varepsilon)_0,$$
$$A_1/A^2 \simeq (A_1/A_\varepsilon)/(A^2/A_\varepsilon) \simeq A_1^\varepsilon/(A_1^\varepsilon)_0, \text{ and}$$
$$G/A^2 \simeq (G/A_\varepsilon)/(A^2/A_\varepsilon).$$

According to a basic theorem concerning the structure of connected compact Lie groups ([2], pp. 144—145) there is an isomorphism

$$\lambda : (T_1 \times \dots \times T_{k-3} \times S)/D \to A^2/A_\varepsilon$$

where T_i, $i = 1, \dots, k-3$ are 1-dimensional toroidal and S a semisimple or 1-dimensional toroidal invariant subgroup of A^2/A_ε, and D is a discrete central subgroup of $T_1\times\cdots\times T_{k-3}\times S$ such that both $(T_1\times\cdots\times T_{k-3})\cap D$ and $S\cap D$ are trivial. Consider now the following decreasing sequence of invariant subgroups of A^2/A_ε:

$$N_i = \lambda\left(\frac{\{e'\}\times\cdots\times\{e'\}\times T_i\times\cdots\times T_{k-3}\times S}{D_i}\right) \quad \text{for} \quad i = 1, \dots, k-3,$$

$$N_{k-2} = S, \quad N_{k-1} = \{e'\},$$

where e' is the identity element and

$$D_i = (\{e'\}\times\cdots\times\{e'\}\times T_i\times\cdots\times T_{k-3}\times S)\cap D.$$

Put now $A^{i+1}=\pi_\varepsilon^{-1}(N_i)$ for $i=1, \dots, k-1$. Then a decreasing sequence of compact invariant subgroups of G is obtained such that A_1/A^2 is finite, A^i/A^{i+1} is a 1-dimensional torus for $i=2, \dots, k-2$, and A^{k-1}/A^k is a compact connected semisimple Lie group or a 1-dimensional torus as the following isomorphisms show:

$$A^{i+1}/A^{i+2} \simeq (A^{i+1}/A_\varepsilon)/(A^{i+2}/A_\varepsilon) \simeq N_i/N_{i+1}.$$

Moreover G/A^{i+1} is a Lie group for $i=1, \dots, k-1$ on account of the isomorphism

$$G/A^{i+1} \simeq (G/A_\varepsilon)/(A^{i+1}/A_\varepsilon).$$

Therefore if A_σ^* for $\sigma\in E=\{1, \dots, k\}\cup\{\alpha|\alpha>\varepsilon, \alpha\in\Delta\}$ is defined by

$$A_1^* = A_1, \quad A_i^* = A^i \quad \text{for} \quad i = 2, \dots, k, \quad A_\sigma^* = A_\sigma \quad \text{for} \quad \sigma > \varepsilon,$$

then the assertions of the lemma obviously hold for this system.

In order to have a short term for the above construction it will be said that the system $\{A_\sigma^*|\alpha\in E\}$ is obtained by *adjusting the system* $\{A_\alpha|\alpha\in\Delta\}$ *up to the index* $\varepsilon\in\Delta$. It is to be noted that $\{A_\alpha|\alpha\in\Delta\}$ and $\varepsilon\in\Delta$ do not define the adjusted system uniquely, the choice and order of the toroidal subgroups being arbitrary to some extent in the construction. In fact this circumstance will be of use yet in later developments.

Another fact which has an important technical role in subsequent arguments is expressed by the following

Lemma 2. *Let G be a Lie group $A\subset G$ a compact invariant subgroup such that G/A is connected and $H\subset G$ a closed subgroup such that with $B=A\cap H$ the group H/B is connected as well. Let C be the centralizer of A in G and D that of B in H. If $\mathfrak{c}, \mathfrak{h}, \mathfrak{d}$ are the Lie algebras of C, H, D then $\mathfrak{d}=\mathfrak{c}\cap\mathfrak{h}$.*

Proof. Let aut $g: G\to G$ be the inner automorphism defined by $g\in G$ and $\mathrm{aut}_A\, g: A\to A$ its restriction to A. Then by an extension of a theorem of K. IWASAWA

([5] and [10]) there is an $a\in A$ such that $\mathrm{aut}_A\, a=\mathrm{aut}_A\, g$ holds. Let C be the centralizer of A in G then $C\cap A$ is the center of A and a continuous isomorphism

$$\varrho: G/C \to A/C\cap A$$

is obtained by setting $\varrho(gC)=a(C\cap A)$ for $g\in G$ and $a\in A$ if and only if $\mathrm{aut}_A\, g= =\mathrm{aut}_A\, a$ holds. If $\mathfrak{g}$, $\mathfrak{a}$, $\mathfrak{c}$ are respectively the Lie algebras of G, A, C then $\mathfrak{c}\cap\mathfrak{a}$ is the Lie algebra of $C\cap A$; moreover $\mathfrak{c}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$ by basic properties of the adjoint representation (see [2], pp. 100—101) and thus $\mathfrak{c}\cap\mathfrak{a}$ is the center of $\mathfrak{a}$. The differential $d\varrho$ of ϱ yields the Lie algebra isomorphism

$$d\varrho: \mathfrak{g}/\mathfrak{c} \to \mathfrak{a}/\mathfrak{c}\cap\mathfrak{a}.$$

Let ad $X: \mathfrak{g}\to\mathfrak{g}$ be the adjoint map defined by X and $\mathrm{ad}_{\mathfrak{a}} X: \mathfrak{a}\to\mathfrak{a}$ its restriction to $\mathfrak{a}$. Then by basic properties of the adjoint representation $\mathrm{ad}_{\mathfrak{a}} X'=\mathrm{ad}\, A'$ for $X'\in X+\mathfrak{c}$ and $A'\in A+(\mathfrak{c}\cap\mathfrak{a})$ if and only if $d\varrho(X+\mathfrak{c})=A+(\mathfrak{c}\cap\mathfrak{a})$ holds. This implies that any coset $X+\mathfrak{c}\in\mathfrak{g}/\mathfrak{c}$ contains an element of $\mathfrak{a}$ and consequently for X the validity of $X\in\mathfrak{a}$ can be assumed without loss of generality. But then $d\varrho(X+\mathfrak{c})=X+(\mathfrak{c}\cap\mathfrak{a})= =(X+\mathfrak{c})\cap\mathfrak{a}$ hold, and this means that $d\varrho$ can be considered as forming the intersection with $\mathfrak{a}$. Let now D be the centralizer of B in H and $\mathfrak{b}$, $\mathfrak{d}$ the Lie algebras of B, D respectively. Then an argument analogous to the preceding one yields the isomorphisms

$$\sigma: H/D \to B/D\cap B \quad \text{and} \quad d\sigma: \mathfrak{h}/\mathfrak{d} \to \mathfrak{b}/\mathfrak{d}\cap\mathfrak{b}$$

with properties analogous to those established above. Consider now the Lie algebra monomorphisms

$$\xi: \mathfrak{h}/\mathfrak{c}\cap\mathfrak{h} \to \mathfrak{g}/\mathfrak{c} \quad \text{and} \quad \eta: \mathfrak{a}\cap\mathfrak{h}/\mathfrak{c}\cap\mathfrak{a}\cap\mathfrak{h} \to \mathfrak{a}/\mathfrak{c}\cap\mathfrak{a}$$

which are defined by the inclusion relation of cosets. Then

$$d\varrho\circ\xi: (\mathfrak{h}/\mathfrak{c}\cap\mathfrak{h}) = \eta(\mathfrak{a}\cap\mathfrak{h}/\mathfrak{c}\cap\mathfrak{a}\cap\mathfrak{h})$$

holds in consequence of the fact that $d\varrho$ can be obtained as intersecting with $\mathfrak{a}$. These imply now that

$$\eta^{-1}\circ d\varrho\circ\xi: \mathfrak{h}/\mathfrak{c}\cap\mathfrak{h} \to \mathfrak{a}\cap\mathfrak{h}/\mathfrak{c}\cap\mathfrak{a}\cap\mathfrak{h}$$

is an isomorphism. In order to show that $\mathfrak{d}\cap\mathfrak{b}=\mathfrak{c}\cap\mathfrak{a}\cap\mathfrak{b}$ holds observe first that $\mathfrak{a}$ as a compact Lie algebra is isomorphic to the direct sum $\mathfrak{s}\oplus(\mathfrak{c}\cap\mathfrak{a})$ where $\mathfrak{b}\subset\mathfrak{a}$ is a semisimple ideal since $\mathfrak{c}\cap\mathfrak{a}$ is the center of $\mathfrak{a}$. Therefore if $Z_i\in b$ and $Z_i=X_i+Y_i$ with $X_i\in\mathfrak{s}$, $Y_i\in\mathfrak{c}\cap\mathfrak{a}$ where $i=1, 2$ then $[Z_1, Z_2]=[X_1, X_2]$. Consequently if Z_1 is fixed then $[Z_1, Z_2]=0$ for every $Z_2\in\mathfrak{b}$ if and only if $X_1=0$, which is equivalent to $Z_1\in\mathfrak{c}\cap\mathfrak{a}$. Thus the isomorphisms

$$d\sigma: \mathfrak{h}/\mathfrak{d} \to \mathfrak{b}/\mathfrak{c}\cap\mathfrak{a}\cap\mathfrak{h} \quad \text{and} \quad \eta^{-1}\circ d\varrho\circ\xi: \mathfrak{h}/\mathfrak{c}\cap\mathfrak{h} \to \mathfrak{b}/\mathfrak{c}\cap\mathfrak{a}\cap\mathfrak{h}$$

together with the obvious $\mathfrak{c}\cap\mathfrak{h}\subset\mathfrak{d}$ yield that $\mathfrak{c}\cap\mathfrak{h}=\mathfrak{d}$ is valid.

The following theorem serves to yield a survey of the local structure of groups approximated by Lie groups. Actually this theorem is a complemented version of a well-known result of IWASAWA [5]. The proof given here is based on ideas due to V. M. GLUŠKOV [10] and works with a concrete approximating inverse system of Lie groups. This way a rather lengthy but perfectly constructive presentation is obtained, a feature essential for the later developments.

Theorem 1. *Let G be a topological group which can be approximated by Lie groups and $H\subset G$ a closed subgroup. Then to any neighborhood U of the identity there is a compact invariant subgroup $A\subset U$ of G and a Lie subgroup $L\subset G$ such that*

1. *$M=L\cap H$ is a Lie subgroup,*
2. *there is a neighborhood $V\subset U$ of the identity in L such that the direct products $V\times A$, $(V\cap H)\times(A\cap H)$ exist and form neighborhoods of the identity in G and H respectively.*

Proof. Let a system $\{A_\alpha|\alpha\in\Delta\}$ of compact invariant subgroups define an inverse system of Lie groups approximating G. If C_1 is the identity component of G_1 then $G'=\pi_1^{-1}(C_1)$ is an open and closed invariant subgroup of G. If C is the centralizer of A_1 in G then $G=CA_1$ by an extension of a theorem of IWASAWA ([5] and [10]). As it is compatible with the above definition of C_1 put $C_\alpha=\pi_\alpha(C)$ for $\alpha\in\Delta$ and let γ_α, γ_α^β be the restrictions of π_α, π_α^β to C, C_β respectively. Thus an inverse system $\{C_\alpha, \gamma_\alpha^\beta\}$ of Lie groups is obtained which approximates C since C is isomorphic to the projective limit $C_\infty=\varprojlim\{C_\alpha|\alpha\in\Delta\}$ under the restriction of $\omega: G\to G_\infty$ to C. The kernel C_α^β of γ_α^β is central in C_β because if

$$\tilde{\gamma}_\beta: C/A_\beta\cap C \to C_\beta$$

is the isomorphism induced by γ_β then the inverse image of C_α^β under $\tilde{\gamma}_\beta$ is given by

$$\tilde{\gamma}_\beta^{-1}(C_\alpha^\beta) = (A_\alpha\cap C)/(A_\beta\cap C)$$

which is obviously central in $C/A_\beta\cap C$.

Analogously let D_1 be the identity component of $H_1=\pi_1(H)$ then $H'=H\cap\cap\pi_1^{-1}(D_1)$ is an open and closed invariant subgroup of H with $H'\subset G'$. Let D be the centralizer of $B_1=A_1\cap H$ in H' then $H'=DB_1$. Put $D_\alpha=\pi_\alpha(D)$ for $\alpha\in\Delta$ and let $\delta_\alpha, \delta_\alpha^\beta$ be the restrictions of $\pi_\alpha, \pi_\alpha^\beta$ to D, D_β respectively. Then an inverse system $\{D_\alpha, \delta_\alpha^\beta\}$ of Lie groups approximating D is obtained, in fact D is isomorphic to the projective limit $D_\infty=\varprojlim\{D_\alpha|\alpha\in\Delta\}$ under the restriction of ω to D.

Set now $G_\alpha'=\pi_\alpha(G')$, $A_\alpha^*=\pi_\alpha(A_1)$, $H_\alpha'=\pi_\alpha(H')$, $B_\alpha^*=\pi_\alpha(B_1)$ and let C_α' be the centralizer of A_α^* in G_α', analogously D_α' the centralizer of B_α^* in H_α' for $\alpha\in\Delta$. Then $C_\alpha\subset C_\alpha'$ and $D_\alpha\subset D_\alpha'$ obviously hold. Let $\gamma_\alpha'^\beta$, $\delta_\alpha'^\beta$ be the restrictions of π_α^β to C_β', D_β' respectively then $\{C_\alpha', \gamma_\alpha'^\beta\}$ are inverse systems of Lie groups and consider their projective limits $C'_\infty = \varprojlim \{C'_\alpha | \alpha \in \Delta\}$ and $D'_\infty = \varprojlim \{D'_\alpha | \alpha \in \Delta\}$. By the above stipulations $C_\infty \subset C'_\infty$ and $D_\infty \subset D'_\infty$ evidently hold, but beyond this even $C_\infty = C'_\infty$ and $D_\infty = D'_\infty$ are valid. In fact ω maps C onto C'_∞ and D onto D'_∞ since C'_∞ is obviously the centralizer of $\omega(A_1)$ in $\omega(G')$ and similarly D'_∞ is the centralizer of $\omega(B_1)$ in $\omega(H')$.

Consider now the Lie algebras $\mathfrak{c}_\alpha$, $\mathfrak{c}'_\alpha$, $\mathfrak{d}_\alpha$, $\mathfrak{d}'_\alpha$ which correspond in due order to the Lie groups C_α, C'_α, D_α, D'_α. Then $\mathfrak{d}_\alpha \subset \mathfrak{c}_\alpha$ is valid for $\alpha \in \Delta$. In fact for $\alpha = 1$ this is a consequence of $D_1 \subset C_1$, on the other hand for $\alpha > 1$ will be verified by the following argument: The isomorphisms

$$G'_\alpha / A^*_\alpha \simeq G'/A_1 \simeq C_1 \quad \text{and} \quad H'_\alpha / B^*_\alpha \simeq H'/B_1 \simeq D_1$$

imply that both G'_α / A^*_α and H'_α / B^*_α are connected. Moreover $B^*_\alpha = A^*_\alpha \cap H'_\alpha$ is valid since $\pi_\alpha(A_1 \cap H') = \pi_\alpha(A_1) \cap \pi_\alpha(H')$ holds. Thus Lemma 2 applies and yields that $\mathfrak{d}'_\alpha \subset \mathfrak{c}'_\alpha$ is true for $\alpha \in \Delta$. Assume now that $\mathfrak{d}_\sigma \nsubseteq \mathfrak{c}_\sigma$ for some $\sigma \in \Delta$. Then there exists an $X^\sigma \in \mathfrak{d}_\sigma$ with $X^\sigma \in \mathfrak{c}'_\sigma - \mathfrak{c}_\sigma$. But in this case $C'_\infty = C_\infty$ implies that there is a $\tau > \sigma$ such that no such $X^\tau \in \mathfrak{c}'_\tau$ exists for which $X^\sigma = d\pi^\tau_\sigma X^\tau$ is true. On the other hand $X^\sigma \in \mathfrak{d}_\sigma$ and the fact that δ^τ_σ is an epimorphism imply the existence of an $X^\tau \in \mathfrak{d}_\tau$ with $X^\sigma = d\pi^\tau_\sigma X^\tau$. Since $\mathfrak{d}_\tau \subset \mathfrak{d}'_\tau \subset \mathfrak{c}'_\tau$, a contradiction is obtained.

Fix now a base $(X^1_1, \ldots, X^1_m, X^1_{m+1}, \ldots, X^1_n)$ of the Lie algebra $\mathfrak{c}_1$ such that $(X^1_{m+1}, \ldots, X^1_n)$ is a base of $\mathfrak{d}_1$ and let c^h_{ij} with $h, i, j = 1, \ldots, n$ be the structural constants of $\mathfrak{c}_1$ with respect to this base. It will be shown now that a system $(X^\alpha_1, \ldots, X^\alpha_n)$ of elements of $\mathfrak{c}_\alpha$ can be chosen simultaneously for every $\alpha \in \Delta$ so as to satisfy the following conditions:

$$X^\alpha_{m+1}, \ldots, X^\alpha_n \in \mathfrak{d}_\alpha,$$

$$X^\alpha_i = d\pi^\beta_\alpha X^\beta_i \quad \text{for} \quad i = 1, \ldots, n \quad \text{and} \quad \alpha, \beta \in \Delta \quad \text{with} \quad \alpha < \beta.$$

In fact a choice of such systems can be carried out by the following transfinite construction: Let $\alpha \in \Delta$ be fixed and assume that such systems have been already selected for each $\xi < \alpha$ so that both the above requirements are fulfilled. The two possibilities that α has or has not an immediate predecessor in Δ have to be considered now apart. In the first case when $\alpha - 1 \in \Delta$ the immediate predecessor of α does exist the required choice of $(X^\alpha_1, \ldots, X^\alpha_n)$ is obviously possible since $\mathfrak{d}_\alpha \subset \mathfrak{c}_\alpha$ holds and $\gamma^\alpha_{\alpha-1}$, $\delta^\alpha_{\alpha-1}$ are surjective. In the second case the fact is helpful that

$$A_\xi / A_\alpha \subset A_\eta / A_\alpha \subset G / A_\alpha \quad \text{for} \quad \eta < \xi < \alpha.$$

This yields the existence of such a $\zeta < \alpha$ that A_ξ / A_α reduces to the identity element of G_α for $\zeta \leqq \xi < \alpha$ and consequently $\pi^\alpha_\xi \colon G_\alpha \to G_\xi$ is an isomorphism for every such index ξ. Thus $X^\xi_i = d\pi^\alpha_\xi X^\alpha_i$ for $i = 1, \ldots, n$ defines the system $(X^\alpha_1, \ldots, X^\alpha_n)$ so as to meet both the above requirements.

Define now $Y_{ij}^{\alpha} \in \mathfrak{c}_{\alpha}$ for $i, j=1, \ldots, n$ and $\alpha \in \Delta$ by

$$Y_{ij} = [X_i, X_j] - \sum_{h=1}^{n} c_{ij}^h X_h.$$

Then $Y_{ij}^{\alpha} = d\pi_{\alpha}^{\beta} Y_{ij}^{\beta}$ obviously holds for $\alpha < \beta$. Moreover all the brackets $[X_h^{\alpha}, Y_{ij}^{\alpha}]$, $[Y_{ij}^{\alpha}, Y_{kl}^{\alpha}]$ vanish because $d\pi_1^{\alpha} Y_{ij}^{\alpha} = Y_{ij}^1 = 0$ implies that Y_{ij}^{α} is an element of the Lie algebra of C_1^{α} and C_1^{α} as the kernel of γ_1^{α} is central in C_{α} by a former observation.

In what next follows an adjustment of the system $\{A_{\alpha} | \alpha \in \Delta\}$ will be carried out in order to provide favourable settings for subsequent steps in the argument. If for a fixed pair (i, j) with $1 \leqq i, j \leqq n$ there is an $\alpha \in \Delta$ with $Y_{ij}^{\alpha} \neq 0$ then there is a first one $\alpha_{ij} > 1$ among such indices. On the other hand if $Y_{ij}^{\alpha} = 0$ for every $\alpha \in \Delta$ then put $\alpha_{ij} = 2$. Consider now $\varepsilon = \max \{\alpha_{ij} | i, j=1, \ldots, n\}$. Since $C_1^{\varepsilon} \subset A_1^{\varepsilon}$ holds the identity component $(C_1^{\varepsilon})_0$ of C_1^{ε} is a subgroup of $(A_1^{\varepsilon})_0$ and it is even central in $(A_1^{\varepsilon})_0$ on account of former stipulations. Moreover the Y_{ij}^{ε} are elements of the Lie algebra of $(C_1^{\varepsilon})_0$. Adopting the notations of Lemma 1 consider A^2/A_{ε} and let $T_1, \ldots, T_{k-3}$ be 1-dimensional toroidal subgroups and S the semisimple or 1-dimensional toroidal subgroup defined there. Put $T_{k-2} = S$ is eventually S is a 1-dimensional torus. Let $\lambda\colon A^2/A_{\varepsilon} \to (A_1^{\varepsilon})_0$ be the canonical isomorphism. Now it is evidently possible to carry out an adjustment of the system $\{A_{\alpha} | \alpha \in \Delta\}$ up to the index ε so as to satisfy even the following two additional requirements:

1. the tangent vectors of $\lambda T_1, \ldots, \lambda T_l$ at the identity form a maximal linearly independent subset of $\{Y_{ij}^{\varepsilon} | i, j=1, \ldots, n\}$ for some $0 \leqq l \leqq k-2$,

2. the tangent vectors of $\lambda T_{l+1}, \ldots, \lambda T_{k-3}$, and of $\lambda S = \lambda T_{k-2}$ if S is a torus, at the identity are not elements of the Lie algebra generated by $\{Y_{ij}^{\varepsilon} | i, j=1, \ldots, n\}$.

Let now $\{\tilde{A}_{\sigma} | \sigma \in E\}$ be the system obtained by this adjustment and consider $\{\tilde{G}_{\sigma}, \tilde{\pi}_{\sigma}^{\tau}\}$ the corresponding approximating inverse system of Lie groups. Then $\tilde{C}_{\sigma}$, $\tilde{D}_{\sigma}$, $\tilde{X}_h^{\sigma}$, $\tilde{Y}_{ij}^{\sigma}$ are obviously uniquely defined for this new system by postulating that $\tilde{C}_{\sigma} = C_{\sigma}$, $\tilde{D}_{\sigma} = D_{\sigma}$, $\tilde{X}_h^{\sigma} = X_h^{\sigma}$, $\tilde{Y}_{ij}^{\sigma} = Y_{ij}^{\sigma}$ hold for $\sigma = 1$ and for every $\sigma \in E$ with $\sigma > k$. Since no possibility of confusion will be caused by this, the tildas will be dropped in denoting quantities corresponding to the adjusted system subsequently.

Consider now the unique Lie algebra $\mathfrak{c}$ which has a base formed by the elements X_h, where $h=1, \ldots, n$ and Y_{ij} where $i, j=1, \ldots, n$ but $i<j$ such that the following relations are satisfied:

$$[X_i, X_j] = \sum_{h=1}^{n} c_{ij}^h X_h + Y_{ij} \quad \text{for} \quad i, j = 1, \ldots, n \quad \text{with} \quad i<j$$

$$[X_h, Y_{ij}] = [Y_{ij}, Y_{st}] = 0 \quad \text{for} \quad h, i, j, s, t = 1, \ldots, n \quad \text{with} \quad i<j,\ s<t.$$

Here the elements X_h, Y_{ij} with $h, i, j = m+1, \ldots, n$ obviously form the basis of a subalgebra $\mathfrak{d} \subset \mathfrak{c}$. Consider the simply connected Lie group P which has $\mathfrak{c}$ as Lie algebra and its connected subgroup Q which corresponds to $\mathfrak{h}$. There is a unique Lie group homomorphism $\varphi_\sigma: P \to C_\sigma$ for every $\sigma \in E$ such that $X_h^\sigma = d\varphi_\sigma X_h$ and $Y_{ij}^\sigma = d\varphi_\sigma Y_{ij}$ for $h, i, j = 1, \dots, n$. Consequently a continuous homomorphism

$$\varphi: P \to C_\infty$$

is defined by setting $\varphi(p) = \{\varphi_\sigma(p) | \sigma \in E\}$ for $p \in P$. Let K be the kernel of φ and $\pi: P \to P' = P/K$ the corresponding canonical epimorphism. Then P' and $Q' = \pi(Q)$ are Lie groups. Therefore if

$$\varphi': P' \to C_\infty$$

is the monomorphism defined by $\varphi = \varphi' \circ \pi$ then $L_\infty = \varphi'(P')$ and $M_\infty = \varphi'(Q')$ are Lie subgroups of G_∞ with $M_\infty \subset L_\infty$ and $M_\infty \subset H_\infty$.

The monomorphism φ' can be obviously given in the form $\varphi'(p') = \{\varphi'_\sigma(p') | \sigma \in E\}$ for $p' \in P'$ where the $\varphi'_\sigma: P' \to C_\sigma$ are Lie group homomorphisms such that $\varphi'_\sigma = \pi_\sigma^\tau \circ \varphi'_\tau$ for $\sigma, \tau \in E$ with $\sigma < \tau$. This implies that the kernel of φ'_τ is a subgroup of the kernel of φ'_σ, consequently there is a first index $\delta \in E$ such that the kernel of φ'_σ is discrete for $\sigma \geqq \delta$. Fix now a left invariant Riemannian metric on P', then there is a unique left invariant Riemannian metric on the Lie group $L_\sigma = \varphi'_\sigma(P')$ for $\sigma \geqq \delta$ such that φ'_σ is a local isometry. Consequently the standard procedure based on the "méthode de rayonnement" due to E. Cartan ([1], pp. 181—186) yields a unique set $F_\sigma^0 \subset P'$ for every $\sigma \in E$ with $\sigma \geqq \delta$ such that

1. F_σ^0 is an open neighborhood of the identity in P',

2. If $\bar{F}_\sigma$ is the closure of F_σ^0 in P' then there is a set F_σ such that $F_\sigma^0 \subset F_\sigma \subset \bar{F}_\sigma$ and the restriction of φ'_σ to F_σ is a continuous bijection onto L_σ.

3. $F_\sigma^0 \subset F_\tau^0$ if $\delta \leqq \sigma \leqq \tau$.

The set F_σ^0 is called the *fundamental domain* of the local isometry φ'_σ. Thus $F_\infty = \varphi'(F_\delta)$ is a neighborhood of the identity in L_∞, if L_∞ is taken with that topology which makes $\varphi': P' \to L_\infty$ an isomorphism. By the preceding stipulations F_∞ and $A_\infty = \omega(A_\delta)$ have a single element in common which is the identity of G_∞. Moreover elements of F_∞ and A_∞ commute on account of the construction. Consider now a compact neighborhood $V' \subset F_\delta^0$ of the identity in P' and $V_\infty = \varphi'(V')$. Then the map

$$V_\infty \times A_\infty \to V_\infty \cdot A_\infty$$

defined by the group multiplication in G_∞ is a continuous bijection. But $V_\infty \times A_\infty$ being compact, this bijection is a homeomorphism. Thus the set $V_\infty \cdot A_\infty$ which contains the identity in G_∞ is isomorphic with the direct product $V_\infty \times A_\infty$.

It has to be shown now that $V_\infty A_\infty$ is a neighborhood of the identity in G_∞, or what amounts to the same thing, that VA with $V = \omega^{-1}(V_\infty)$, $A = \omega^{-1}(A_\infty)$ is a neighborhood of the identity in G. Evidently it suffices to prove that $V_\delta = \varphi'_\delta(V')$ is a neighborhood of the identity in G_δ since then by $VA = \pi_\delta^{-1}(V_\delta)$ the assertion follows. In order to carry this out the value of the index δ will be elicited in what follows. Consider first the case when every Y_{ij}^{α} does vanish with respect to the original approximating inverse system defined by $\{A_\alpha|\alpha\in\Delta\}$. Then every Y_{ij}^{σ} vanishes with respect to the adjusted system as well, and consequently $\delta=1$ holds. But then V_1 is evidently a neighborhood of the identity in G_1. If there is a $Y_{ij}^{\alpha}\neq 0$ with respect to the original system then $Y_{ij}^{k}\neq 0$ obviously holds with respect to the adjusted one. This implies that $l\geqq 1$ and it will be shown now that $\delta=l+1$ holds. Observe first that the kernel of φ_k is discrete by the definition of ε and in consequence of $A_k^*=A_\varepsilon$. Thus the kernel of φ_k' is discrete as well. Assume now that the kernel of φ_{l+1}' is not discrete. Then there exists a non-zero $Z'\in\mathfrak{c}'$ with $d\varphi_{l+1}'(Z')=0$ where $\mathfrak{c}'$ is the Lie algebra of P'. Therefore $d\varphi_{l+1}'(Z')=d\pi_{l+1}^{k}\circ d\varphi_k'(Z')=0$ and $Z^k=d\varphi_k'(Z')\neq 0$ yield that Z^k is an element of the Lie algebra of the kernel of π_{l+1}^{k}. But then Z^k cannot be expressed as a linear combination of the vectors $\{Y_{ij}^{k}|i,j=1,\dots,n\}$ on account of the definition of l. On the other hand such an expression does exist since Z' is a linear combination of the $Y_{ij}'=d\pi(Y_{ij})$, $i,j=1,\dots,n$. This contradiction shows that the kernel of φ_{l+1}' must be discrete. Consider now that Y_{st}^{k} which is tangent vector at the identity of the 1-dimensional toroidal subgroup T_l of A^2/A_ε. Then there is a non-zero $Y_{st}'\in\mathfrak{c}'$ such that $Y_{st}^{k}=d\varphi_k'(Y_{st}')$ holds. But then $d\varphi_l'(Y_{st}')=$ $=d\pi_l^k\circ d\varphi_k'(Y_{st}')=0$ shows that the kernel of φ_l' cannot be discrete. Consequently $\delta=l+1$ is valid. Now in order to verify that V_{l+1} is a neighborhood of the identity in G_{l+1} it is obviously sufficient to note that by the construction of the adjusted system the Lie algebra of the kernel of π_1^{l+1} is spanned by the vectors $\{Y_{ij}^{l+1}|i,j=1,\dots,n\}$ and that V_1 is a neighborhood of the identity in G_1.

The set $V_\infty\cap M_\infty$ is a neighborhood of the identity in M_∞ since $V'\cap Q'$ is a neighborhood of the identity in Q' and $\varphi'(V'\cap Q')=\varphi'(V')\cap\varphi'(Q')=V_\infty\cap M_\infty$. Moreover $M_\infty=L_\infty\cap H_\infty$ holds on account of the construction and thus $V_\infty\cap M_\infty=$ $=V_\infty\cap H_\infty$ implies that the direct product

$$(V_\infty\cap H_\infty)\times(A_\infty\cap H_\infty)$$

as isomorphic to the set $(V_\infty\cap H_\infty)(A_\infty\cap H_\infty)$. In order to prove that this set is a neighborhood of the identity in H_∞, or what ammounts to the same fact, $(V\cap H)(A\cap H)$ is a neighborhood of the identity in H, it is sufficient to show that $VA\cap H\subset(V\cap H)(A\cap H)$ since the converse is obvious. But $h\in VA\cap H$ implies that $\pi_\delta(h)\in\pi_\delta(VA\cap H)\subset\pi_\delta(VA)\cap\pi_\delta(H)=V_\delta\cap H_\delta=\varphi_\delta'(V'\cap Q')$ holds. Consequently there exist $v\in V\cap H$ and $a\in A$ such that $h=va$. Thus $a=v^{-1}h$ yields that $a\in H$ and therefore $a\in A\cap H$.

The subgroup A_δ, $\delta\in E$ which has a cruical role in the above proof will yet occure repeatedly at decisive steps of some subsequent arguments. In order to provide a short term it will be called the *locally factorizing element of the adjusted system* $\{A_\sigma|\sigma\in E\}$.

The following lemma is an easy consequence of a well-known theorem concerning the structure of connected compact groups ([8], pp. 88—93). However a proof is given here for convenience since some facts and objects occurring in the argument will be yet of use later on.

Lemma 3. *Let G be a connected compact group then there is a connected compact group $\bar{G}$ and a continuous epimorphism $\lambda\colon \bar{G}\to G$ such that to any connected closed subgroup $H\subset G$ there exists a connected closed subgroup $\bar{H}\subset\bar{G}$ with $H=\lambda(\bar{H})$, which is invariant in and even a direct factor of $\bar{G}$ provided that H is invariant in G.*

Proof. Let $\{G_\alpha, \pi_\alpha^\beta\}$ be an inverse system of Lie groups approximating G. Then every G_α is a connected compact group and thus by the structural theorem of such groups there are connected closed invariant subgroups T_α, $S_\alpha\subset G_\alpha$ and a continuous epimorphism

$$\mu_\alpha\colon T_\alpha\times S_\alpha\to G_\alpha$$

such that T_α is central, S_α is semisimple and the kernel of μ_α is discrete. Since $T_\alpha=$ $=\pi_\alpha^\beta(T_\beta)$ and $S_\alpha=\pi_\alpha^\beta(S_\beta)$ are obviously valid for $\alpha<\beta$, the inverse system $\{T_\alpha\times S_\alpha\, \hat{\pi}_\alpha^\beta\}$ can be formed where $\bar{\pi}_\alpha^\beta=\pi_\alpha'^\beta\times\pi_\alpha''^\beta$ and $\pi_\alpha'^\beta$, $\pi_\alpha''^\beta$ are the restrictions of π_β^α to T_β, S_β respectively. Let $\bar{S}_\alpha$ be the universal covering group of S_α and $\varkappa_\alpha\colon \bar{S}_\alpha\to S_\alpha$ the covering epimorphism, then there is a unique lift $\sigma_\alpha^\beta\colon \bar{S}_\beta\to\bar{S}_\alpha$ of $\pi_\alpha''^\beta$ such that

$$\begin{array}{ccc}
\bar{S}_\alpha & \xleftarrow{\;\sigma_\alpha^\beta\;} & \bar{S}_\beta \\
\downarrow{\scriptstyle\varkappa_\alpha} & & \downarrow{\scriptstyle\varkappa_\beta} \\
S_\alpha & \xleftarrow[\;\pi_\alpha''^\beta\;]{} & S_\beta
\end{array}$$

Consequently the inverse system $\{\bar{G}_\alpha, \bar{\pi}_\alpha^\beta\}$ can be formed where $\bar{G}_\alpha=T_\alpha\times\bar{S}_\alpha$ and $\bar{\pi}_\alpha^\beta=\pi_\alpha'^\beta\times\sigma_\alpha^\beta$. Moreover the commutative diagrams

$$\begin{array}{ccc}
\bar{G}_\alpha & \xleftarrow{\;\bar{\pi}_\alpha^\beta\;} & \bar{G}_\beta \\
\downarrow{\scriptstyle\nu_\alpha} & & \downarrow{\scriptstyle\nu_\beta} \\
T_\alpha\times S_\alpha & \xleftarrow{\;\hat{\pi}_\alpha^\beta\;} & T_\beta\times S_\beta \\
\downarrow{\scriptstyle\mu_\alpha} & & \downarrow{\scriptstyle\mu_\beta} \\
G_\alpha & \xleftarrow[\;\pi_\alpha^\beta\;]{} & G_\beta
\end{array}$$

are obtained where ν_α is the direct product of the identity epimorphism on T_α and of $\varkappa_\alpha$. Put $\lambda_\alpha=\mu_\alpha \circ \nu_\alpha$ then the system $\{\lambda_\alpha|\alpha\in\Delta\}$ of epimorphisms obviously defines a continuous epimorphism

$$\lambda_\infty: \varprojlim \{\bar{G}_\alpha|\alpha\in\Delta\} \to \varprojlim \{G_\alpha|\alpha\in\Delta\}$$

of the projective limits on account of the commutativity of the preceding diagram. Let now $\omega\colon G\to\varprojlim \{G_\alpha|\alpha\in\Delta\}$ be the canonical isomorphism defined by the approximation and put $\bar{G}=\varprojlim \{\bar{G}_\alpha|\alpha\in\Delta\}$ then

$$\omega^{-1}\circ\lambda_\infty = \lambda\colon \bar{G}\to G$$

is a continuous epimorphism and $\bar{G}$ is obviously connected and compact.

Consider now a connected closed subgroup $H\subset G$ then $H_\alpha=\pi_\alpha(H)$ is a connected closed subgroup of G_α. Therefore on account of facts already mentioned in the proof of Lemma 2 there are connected closed invariant subgroups T_α^H, $S_\alpha^H\subset H_\alpha$ with $T_\alpha^H\subset T_\alpha$, $S_\alpha^H\subset S_\alpha$ such that μ_α maps the subgroup $T_\alpha^H\times S_\alpha^H$ of $T_\alpha\times S_\alpha$ onto H_α. Let $\bar{S}_\alpha^H$ be the identity component of $\varkappa_\alpha^{-1}(S_\alpha^H)$ then λ_α maps $\bar{H}_\alpha=T_\alpha^H\times\bar{S}_\alpha^H$ onto H_α. Consequently $\bar{H}=\varprojlim \{\bar{H}_\alpha|\alpha\in\Delta\}$ is a connected closed subgroup of $\bar{G}$ and $H=\lambda(\bar{H})$ is valid.

Assume now that A is a connected closed invariant subgroup of G. Then the preceding construction applied to A yields a connected closed subgroup $\bar{A}$ of $\bar{G}$ with $A=\lambda(\bar{A})$. It is easy to see that $\bar{A}_\alpha$ is an invariant subgroup of $\bar{G}_\alpha$ and this implies that $\bar{A}$ is invariant in $\bar{G}$. Moreover the fact that $\bar{A}_\alpha$ is invariant in $\bar{G}_\alpha$ and the construction of $\bar{G}_\alpha$ implies that $\bar{A}_\alpha$ is a direct factor of $\bar{G}_\alpha$ consequently there is a connected closed invariant subgroup $\bar{B}_\alpha\subset\bar{G}_\alpha$ with $\bar{G}_\alpha=\bar{A}_\alpha\times\bar{B}_\alpha$. Since the epimorphisms λ_α have discrete kernel and by the commutativity of the diagram above $\bar{A}_\alpha=\bar{\pi}_\alpha^\beta(\bar{A}_\beta)$ and $\bar{B}_\alpha=\bar{\pi}_\alpha^\beta(\bar{B}_\beta)$ are obviously valid, and thus the inverse systems $\{\bar{A}_\alpha, \bar{\pi}_\alpha'^\beta\}$ $\{\bar{B}_\alpha, \bar{\pi}_\alpha''^\beta\}$ can be formed where $\bar{\pi}_\alpha'^\beta$, $\bar{\pi}_\alpha''^\beta$ are the corresponding restrictions of $\bar{\pi}_\alpha^\beta$. Consequently $\bar{G}=\bar{A}\times\bar{B}$ is valid where $\bar{B}=\varprojlim \{\bar{B}_\alpha|\alpha\in\Delta\}$.

Corollary. *Let G be a connected compact group, $H\subset G$ a connected closed subgroup, $A\subset G$ a connected closed invariant subgroup and $\bar{A}$, $\bar{B}$, $\bar{H}\subset\bar{G}$ the groups which respectively correspond to them by the preceding constructions. Then $\bar{H}=\bar{A}_H\times\bar{B}_H$ holds where $\bar{A}_H$, $\bar{B}_H$ are connected closed invariant subgroups of $\bar{H}$ with $\bar{A}_H\subset\bar{A}$, $\bar{B}_H\subset\bar{B}$.*

Proof. Abiding by the settings of the above proof consider $\bar{A}_\alpha^H$ the identity component of $\lambda_\alpha^{-1}(A_\alpha\cap H_\alpha)$. Then $\bar{A}_\alpha^H$ is a connected closed invariant subgroup of $\bar{H}_\alpha$, and consequently it is a direct factor of $\bar{H}_\alpha=T_\alpha^H\times\bar{S}_\alpha^H$ since $\bar{S}_\alpha^H$ is simply connected closed invariant subgroup $\bar{B}_\alpha^H$ of $\bar{H}_\alpha$ with $\bar{H}_\alpha=\bar{A}_\alpha^H\times\bar{B}_\alpha^H$ and $\bar{B}_\alpha^H\subset\bar{B}_\alpha$. Thus $\bar{H}=\bar{A}_H\times\bar{B}_H$ is valid with $\bar{A}_H=\varprojlim \{\bar{A}_\alpha^H|\alpha\in\Delta\}$, $\bar{B}_H=\varprojlim \{\bar{B}_\alpha^H|\alpha\in\Delta\}$.

2. The structure of coset spaces of locally compact groups

Let G be a locally compact group, $H \subset G$ a closed subgroup and $\chi: G \to G/H$ the canonical projection. If $A \subset G$ is a compact invariant subgroup and $\pi: G \to G' = G/A$ the canonical epimorphism then $H' = \pi(H)$ is a closed subgroup of G'. Let now $\chi': G' \to G'/H'$ be the canonical projection then there is a unique map $\varphi: G/H \to G'/H'$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\ \chi\ } & G/H \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ G' & \xrightarrow[\ \chi'\ ]{} & G'/H' \end{array}$$

is commutative. The map φ which is continuous and open generates a fiber structure on the space G/H. Since the terminology of fiber structures will prove convenient subsequently, the map φ will be called the *fiber structure defined by the invariant subgroup A on the coset space G/H.*

The result of Montgomery and Zippin on the characterization of transitive Lie group actions is based on the fact that if G/H is a finite dimensional coset space of a group G which can be approximated by Lie groups then there exists a compact invariant subgroup A of G defining a locally trivial fiber structure φ on G/H such that the base space G'/H' is a manifold and the fibers are totally disconnected ([6], pp. 236—246 and [11]). In what next follows this theorem is generalized for arbitrary coset spaces of locally compact groups. At first that case when G can be approximated by Lie groups will be settled by

Lemma 4. *Let H be a closed subgroup of the group G which can be approximated by Lie groups. Then a locally factorizing invariant subgroup A of G defines a locally trivial fiber structure $\varphi: G/H \to G'/H'$ such that the base space G'/H' is a manifold and the fibers are homeomorphic to the coset space A/B where $B = A \cap H$.*

Proof. Let A be a locally factorizing invariant subgroup of G. In order to show that the fiber structure φ defined by A is locally trivial consider the Lie subgroups L, M of G and the neighborhoods V, $V \cap H$ of the identity in these subgroups which correspond to A according to Theorem 1. Let S be a symmetric open neighborhood of the identity in L such that $S^2 \subset V$. Put $T = S \cap M$ then there is a cell $Z \subset S$ and a homeomorphism

$$\gamma: Z \times T \to S$$

where $Z\times T$ is a cartesian product only but γ is given by $\gamma(z, t)=zt$. Let now

$$\alpha\colon S\times A \to SA \quad\text{and}\quad \beta\colon T\times B \to TB$$

be the isomorphisms of the direct products which exist according to Theorem 1; thus $\alpha(s, a)=sa$ and $\beta(t, b)=tb$. Since $\bar{S}=SA$ is a neighborhood of the identity in G, thet set $S'=\pi(\bar{S})$ is a neighborhood of the identity in $G'=G/A$. Moreover $\bar{Z}=\chi(\bar{S})$ is a neighborhood of H in G/H and $Z'=\chi'(S')=\varphi(\bar{Z})$ is a neighborhood of H' in G'/H'. Let now

$$\varkappa\colon S \to S\times A \quad\text{and}\quad \lambda\colon Z \to T$$

be the embeddings given by the trivial cross-sections through the identities, then

$$\pi\circ\alpha\circ\varkappa\colon S \to S' \quad\text{and}\quad \mu = \chi'\circ\pi\circ\alpha\circ\varkappa\circ\gamma\circ\lambda\colon Z \to Z'$$

are obviously homeomorphisms. Consider now the canonical projection

$$\xi\colon Z\times T\times A \to Z\times A/B.$$

It will be shown now that there exists a homeomorphism

$$\eta\colon Z\times A/B \to \bar{Z}$$

which is uniquely defined by the requirement that the diagram

$$\begin{array}{ccc} Z\times T\times A & \xrightarrow{\ \xi\ } & Z\times A/B \\ \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\eta} \\ \bar{S} & \xrightarrow[\ \chi\ ]{} & \bar{Z} \end{array}$$

be commutative. In fact consider $(z_i, t_i, a_i)\in Z\times T\times A$ and $g_i=s_i a_i=z_i t_i a_i$ where $i=1, 2$ such that $\chi(g_1)=\chi(g_2)$ holds. Thus there is an $h\in H$ with $g_2=g_1 h$, and consequently $h=g_1^{-1}g_2=(s_1 a_1)_1^{-1} s_2 a_2=s_1^{-1}s_2 a_1 a_2\in S^2A\subset VA$. Hence $h\in VA\cap H$ and therefore $h=tb$ with $t\in T$, $b\in B$ according to Theorem 1. Thus $g_2=z_2 t_2 a_2=g_1 h=$ $=z_1 t_1 a_1 tb=z_1 t_1 t\, ab$ which imply that

$$z_2 = z_1, \quad t_2 = t_1 t \quad\text{and}\quad a_2 = a_1 b.$$

Conversely the validity of these equalities obviously implies that $\chi(g_1)=\chi(g_2)$ holds. The preceding assertions yield now that

$$\alpha^{-1}\circ\chi^{-1}(\bar{z}) = \{z\}\times T\times aB$$

for $\bar{z}\in\bar{Z}$ with $z\in Z$ and $a\in A$. Therefore the existence and uniqueness of η with the above required properties obviously follows.

In order to prove now that the fiber structure φ is locally trivial at $\chi(H)\in G/H$ consider the homeomorphism

$$\Phi: Z'\times A/B \to Z$$

given by $\Phi(z', aB)=zaH$ where $z=\mu^{-1}(z')$. Then $\varphi\circ\Phi(z', aB)=\varphi(zaH)=\chi'\circ\circ\pi(za)=z'$ hold by the preceding stipulations.

The fact that the fiber structure φ is locally trivial at $\chi(g)=gH\in G/H$ can be shown evidently by means of the homeomorphisms

$$\Lambda_g: G/H \to G/H \quad \text{and} \quad \Lambda_{g'}: G'/H' \to G'/H'$$

which are defined by the left translations $L_g\colon G\to G$ and $L_{g'}\colon G'\to G'$ where $g'=\pi(g)$. Now $\bar{Z}_g=\Lambda_g(\bar{Z})$ is a neighborhood of gH in G/H and $Z'_{g'}=\Lambda_{g'}(Z')$ that of $g'H'$ in G'/H'. Moreover $\mu_g\colon Z\to Z'_{g'}$ defined by $\mu_g=\Lambda_{g'}\circ\mu$ is a homeomorphism. Consider now the map

$$\Phi_g: Z_{g'}\times A/B \to \bar{Z}_g$$

defined by $\Phi_g(g'z', aB)=gzaH$ where $z=\mu_g^{-1}(g'z')$. Then $\varphi\circ\Phi_g(g'z', aB)=g'z'$and Φ_g is obviously a homeomorphism.

The proof of the assertions concerning the base space and the fibers are implicitely contained in the above argument.

The extension of the above results to locally compact groups in entire generality will be carried out by a standard method [11] based on the following lemma, the proof of which being a prerequisite for subsequent considerations is reproduced here for convenience.

Lemma 5. *Let G^* be an open and H a closed subgroup of the topological group G. Then the coset space G/H is the free union of its subsets which are homeomorphic to coset spaces G^*/H^* where $H^*=G^*\cap gHg^{-1}$ with some $g\in G$.*

Proof. The sets G^*gH for $g\in G$ are obviously open in G, and since two such sets are either identical or disjoint they are closed in G as well. Consequently there is an index set Λ and to any $\alpha\in\Lambda$ an element $g_\alpha\in G$ such that

$$G=\cup\{G^*g_\alpha H|\alpha\in\Lambda\}$$

where $G^*g_\alpha H$ and $G^*g_\beta H$ are disjoint if $\alpha\neq\beta$. Let $\chi\colon G\to G/H$ be the canonical projection then the sets $\chi(G^*g_\alpha H)$ are both open and closed in G/H and $\chi(G^*g_\alpha H)$,

$\chi(G^* g_\beta H)$ are disjoint if $\alpha \neq \beta$. Consequently G/H is the free union of these sets. Let now

$$\Psi_\alpha : G^* \to \chi(G^* g_\alpha H)$$

be defined by $\psi_\alpha(g) = \chi(g g_\alpha)$ for $g \in G^*$ then ψ_α is surjective. Moreover $\psi_\alpha(g_1) = \psi_\alpha(g_2)$ if and only if $g_1 g_\alpha H = g_2 g_\alpha H$ which is equivalent to $g_1^{-1} g_2 \in g_\alpha H g_\alpha^{-1}$. Consequently ψ_α is a canonical projection onto the coset space G^*/H_α^* where $Hg_\alpha^* = G^* \cap \cap g_\alpha Hg$. Therefore $\chi(G^* g_\alpha H)$ is homeomorphic to G^*/Hg_α^*.

According to a well-known theorem of H. YAMABE [9] a locally compact group G always has an open subgroup G^* which can be approximated by Lie groups. Therefore if $H \subset G$ is a closed subgroup then G/H is the free union of coset spaces of G^* by the preceding Lemma. Since an invariant subgroup A of G^* defines a fiber structure φ_α on G^*/H_α^* a fiber structure is obtained on $\chi(G^* g_\alpha H)$. There is a unique extension φ of all these φ_α, $\alpha \in A$ on G/H which obviously yields a fiber structure $\varphi : G/H \to G'/H'$. As in general A is not an invariant subgroup of G, this fiber structure φ is not defined by A in the above specified sense. Accordingly it will be said that $\varphi : G/H \to G'/H'$ is a *fiber structure corresponding to the invariant subgroup A of G^**. The preceding two lemmas obviously yield the following

Theorem 2. *Let G be a locally compact group $H \subset G$ a closed subgroup and $G^* \subset G$ an open subgroup which can be approximated by Lie groups. If A is a locally factorizing invariant subgroup of G^* and $\varphi : G/H \to G'/H'$ a fiber structure corresponding to A, then φ is locally trivial, the base space G'/H' is a free union of manifolds and the fibers are homeomorphic to coset spaces of A.*

Strictly speaking the above theorem is not a generalization of the one due to Montgomery and Zippin concerning finite dimensional coset spaces of locally compact groups. This is easily seen from the fact that the assertion about the fiber type in the above theorem does not reproduce that of Montgomery and Zippin by assuming G/H finite dimensional. In solving the problem considered here, however, Theorem 2 has a role analogous to that of the theorem due to Montgomery and Zippin in solving this problem in the special finite dimensional case.

A result of A. BOREL ([3], pp. 306—310) implies that if G is a compact group and $H \subset G$ a closed subgroup such that G/H is contractible then $H = G$ holds. The following lemma extends the validity of this assertion.

Lemma 6. *Let G be a compact group, $H \subset G$ a closed subgroup, $\chi : G \to G/H$ the canonical projection, $A \subset G$ a closed invariant subgroup and $A' = \chi(A)$. If A' is contractible over G/H then $A \subset H$.*

Proof. Let A_0, $H_0 \subset G_0$ be respectively the identity components of A, $H \subset G$ and consider the connected compact group $\bar{G}$, the epimorphism $\lambda : \bar{G} \to G_0$ and the connected closed subgroups $\bar{A}$, $\bar{H}\subset\bar{G}$ with $A_0=\lambda(\bar{A})$ and $H_0=\lambda(\bar{H})$ given by Lemma 3. Thus $\lambda(\bar{A}_H)=\lambda(\bar{H}\cap\bar{A})\subset\lambda(\bar{H})\cap\lambda(\bar{A})=H_0\cap A_0$ is valid. Consequently there are unique continuous surjections $\xi:\bar{G}/\bar{H}\to G_0/H_0$ and $\eta:\bar{A}/\bar{A}_H\to A_0/H_0\cap A_0$ such that

$$\begin{array}{ccc} \bar{G} & \xrightarrow{\ \bar{\chi}\ } & \bar{G}/\bar{H} \\ {\scriptstyle\lambda}\big\downarrow & & \big\downarrow{\scriptstyle\xi} \\ G_0 & \xrightarrow[\ \chi_0\ ]{} & G_0/H_0 \end{array} \qquad\qquad \begin{array}{ccc} \bar{A} & \xrightarrow{\ \bar{\chi}^A\ } & \bar{A}/\bar{A}_H \\ {\scriptstyle\lambda^A}\big\downarrow & & \big\downarrow{\scriptstyle\eta} \\ A_0 & \xrightarrow[\ \chi_0^A\ ]{} & A_0/H_0\cap A_0 \end{array}$$

where λ^A is the restricted epimorphism and $\bar{\chi}$, χ_0, $\bar{\chi}^A$, χ_0^A are the canonical projections. The subgroup $\bar{A}$ is a direct factor of $\bar{G}$ according to Lemma 3 since A_0 is invariant in G_0. Thus there is a closed connected invariant subgroup $\bar{B}\subset\bar{G}$ with $\bar{G}=\bar{A}\times\bar{B}$. Moreover $\bar{H}=\bar{A}_H\times\bar{B}_H$ with the subgroups $\bar{A}_H$, $\bar{B}_H$ given by the corollary of the same lemma. Thus the coset space $\bar{G}/\bar{H}=(\bar{A}\times\bar{B})/(\bar{A}_H\times\bar{B}_H)$ can and in what follows will by identified with the cartesian product $(\bar{A}/\bar{A}_H)\times(\bar{B}/\bar{B}_H)$. Let now

$$\bar{\varepsilon}:\bar{A}/\bar{A}_H\to(\bar{A}/\bar{A}_H)\times(\bar{B}/\bar{B}_H)$$

be the embedding defined by $\bar{\varepsilon}(\bar{a}\bar{A}_H)=(\bar{a}\bar{A}_H,\bar{e}\bar{B}_H)$ where $\bar{e}\in\bar{G}$ is the identity element. Consider moreover the homeomorphism $\varepsilon_0: A_0/H_0\cap A_0\to\chi_0(A_0)$ which is uniquely defined by the validity of $\varepsilon_0\circ\chi_0^A=\chi_0$. Thus the following commutative diagram is obtained

$$\begin{array}{ccc} \bar{A}/\bar{A}_H & \xrightarrow{\ \bar{\varepsilon}\ } & (\bar{A}/\bar{A}_H)\times(\bar{B}/\bar{B}_H)=\bar{G}/\bar{H} \\ {\scriptstyle\eta}\big\downarrow & & \big\downarrow{\scriptstyle\xi} \\ A_0/H_0\cap A_0 & \xrightarrow[\ \varepsilon_0\ ]{} & G_0/H_0 \end{array}$$

In fact the validity of $\xi\circ\bar{\varepsilon}(\bar{a}\bar{A}_H)=\xi(\bar{a}\bar{A}_H,\bar{e}\bar{B}_H)=\xi(\bar{a}\bar{H})=\xi\circ\bar{\chi}(\bar{a})=\chi_0\circ\lambda(\bar{a})$ and of $\varepsilon_0\circ\eta(\bar{a}\bar{A}_H)=\varepsilon_0\circ\eta\circ\bar{\chi}^A(\bar{a})=\varepsilon_0\circ\chi_0^A\circ\lambda^A(\bar{a})=\chi_0\circ\lambda(\bar{a})$ implies the commutativity of this diagram. The inclusions $H_0\subset H\cap G_0$ and $H_0\cap A_0\subset H\cap A_0$ canonically define the continuous surjections

$$\mu: G_0/H_0\to G_0/H\cap G_0 \quad\text{and}\quad \nu: A_0/H_0\cap A_0\to A_0/H\cap A_0$$

which are obviously covering maps. Moreover if

$$\chi_+: G_0\to G_0/H\cap G_0 \quad\text{and}\quad \chi_+^A: A_0\to A_0/H\cap A_0$$

are the canonical projections then there is a homeomorphism $\varepsilon_+\colon A_0/H\cap A_0\to\chi_+(A_0)$ which is uniquely defined by the validity of $\varepsilon_+\circ\chi_+^A=\chi_+$. Hence the following commutative diagram is obtained

$$\begin{array}{ccc} A_0/H_0\cap A_0 & \xrightarrow{\ \varepsilon_0\ } & G_0/H_0 \\ \Big\downarrow{\scriptstyle\nu} & & \Big\downarrow{\scriptstyle\mu} \\ A_0/H\cap A_0 & \xrightarrow[\ \varepsilon_+\ ]{} & G_0/H\cap G_0 \end{array}$$

In fact the validity of $\mu\circ\varepsilon_0(a(H_0\cap A_0))=\mu(aH_0)=a(H\cap G_0)$ and of $\varepsilon_+\circ\nu(a(H_0\cap A_0))=\varepsilon_+(a(H\cap A_0))=\varepsilon_+\circ\chi_+^A(a)=\chi_+(a)=a(H\cap G_0)$ implies that the diagram commutes. Let now $\chi^A\colon A\to A/H\cap A$ be the canonical projection then there is a homeomorphism $\varepsilon\colon A/H\cap A\to A'=\chi(A)$ which is uniquely defined by the validity of $\varepsilon\circ\chi^A=\chi$. The results obtained in proving Lemma 5 yield the homeomorphisms $\varphi\colon G_0/H\cap G_0\to\chi(G_0)$ and $\psi\colon A_0/H\cap A_0\to\chi^A(A_0)$ such that $\chi(g)=\varphi\circ\chi_+(g)$ for $g\in G_0$ and $\chi^A(a)=\psi\circ\chi_+^A(a)$ for $a\in A_0$. Moreover these results yield that $\chi^A(A_0)$ is a component of $A/H\cap A$. But the assumption that A' is contractible over G/H implies that A' is connected. Consequently the map $\psi\circ\nu\circ\eta$ is surjective. Thus the following commutativ diagram is obtained

$$\begin{array}{ccc} A_0/H\cap A_0 & \xrightarrow{\ \varepsilon_+\ } & G_0/H\cap G_0 \\ \Big\downarrow{\scriptstyle\psi} & & \Big\downarrow{\scriptstyle\varphi} \\ A/H\cap A & \xrightarrow[\ \varepsilon\ ]{} & G/H \end{array}$$

In fact the validity of $\varphi\circ\varepsilon_+(a(H\cap A_0))=\varphi\circ\varepsilon_+\circ\chi_+^A(a)=\varphi\circ\chi_+(a)=\chi(a)$ and of $\varepsilon\circ\psi(a(H\cap A_0))=\varepsilon\circ\psi\circ\chi_+^A(a)=\varepsilon\circ\chi^A(a)=\chi(a)$ implies that the diagram commutes.

The assumption that A' is contractible over G/H obviously implies that there exists a continuous map

$$\varkappa\colon A/H\cap A\times I\to G/H,\quad I=[0,1]$$

which is a deformation of the imbedding ε into a constant map of $A/H\cap A$ into G/H, in other words $\varkappa(x,0)=\varepsilon(x)$ for $x\in A/H\cap A$, and $\varkappa(x,1)$ is the same point of G/H for every $x\in A/H\cap A$. Moreover by the above stipulations $\varphi\circ\mu$ is a covering map of G_0/H_0 onto $\chi(G_0)$, and since $(\varphi\circ\mu)\circ\varepsilon_0=\varepsilon\circ\psi\circ\nu$ holds the map $\varepsilon\circ\psi\circ\nu$ is covered by ε_0. Furthermore the continuous map

$$\varkappa': A_0/H_0\cap A_0\times I \to G/H$$

which is defined by $\varkappa'(x, \tau)=\varkappa(\psi\circ\nu(x), \tau)$ for $x\in A_0/H_0\cap A_0$ and $\tau\in I$ is obviously a deformation of $\varepsilon\circ\psi\circ\nu$ into a constant map. Consequently there exists a lift

$$\varkappa_0: A_0/H_0\cap A_0\times I \to G_0/H_0$$

of $\varkappa'$ which is a homotopy of ε_0; in other words $\varkappa_0$ is a continuous map such that $\varphi\circ\mu\circ\varkappa_0(x, \tau)=\varkappa'(x, \tau)$ and $\varkappa_0(x, 0)=\varepsilon_0(x)$ for $x\in A_0/H_0\cap A_0$, $\tau\in I$. Thus $\varkappa_0$ is obviously a homotopy from ε_0 to a constant map. Observe now that the map $\bar{\varepsilon}$ is covering $\varepsilon_0\circ\eta$ since $\xi\circ\bar{\varepsilon}=\varepsilon_0\circ\eta$ by the commutativity of the diagram in which they occur, and that the continuous map

$$\varkappa_0': \bar{A}/\bar{A}_H\times I \to G_0/H_0$$

which is defined by $\varkappa_0'(x, \tau)=\varkappa_0(\eta(x), \tau)$ for $x\in\bar{A}/\bar{A}_H$, $\tau\in I$ is a homotopy from $\varepsilon_0\circ\eta$ to a constant map. It will be shown now that even this homotopy $\varkappa_0'$ can be lifted. Consider for this purpose a system of invariant subgroups of G_0 which defines a system $\{G_\alpha, \pi_\alpha^\beta\}$ of Lie groups approximating G_0. Then the construction made in proving Lemma 3 yields the inverse system $\{\bar{G}_\alpha, \bar{\pi}_\alpha^\beta\}$ of Lie groups approximating $\bar{G}$ along with the system $\{\lambda_\alpha|\alpha\in\Delta\}$ of epimorphisms and with the closed subgroups $H_\alpha\subset G_\alpha$, $\bar{H}_\alpha\subset\bar{G}_\alpha$. Then since $H_\alpha=\lambda_\alpha(\bar{H}_\alpha)$ is valid the epimorphism λ_α defines a continuous surjection ξ_α: $\bar{G}_\alpha/\bar{H}_\alpha\to G_\alpha/H_\alpha$ such that

$$\begin{array}{ccc}
\bar{G}_\alpha & \xrightarrow{\bar{\chi}_\alpha} & \bar{G}_\alpha/\bar{H}_\alpha \\
\downarrow{\scriptstyle\lambda_\alpha} & & \downarrow{\scriptstyle\xi_\alpha} \\
G_\alpha & \xrightarrow[\chi_\alpha]{} & G_\alpha/H_\alpha
\end{array}$$

where χ_α, $\bar{\chi}_\alpha$ are the canonical projections. Since the epimorphism λ_α has discrete kernel ξ_α is a covering map. On account of the fact that $H_\alpha=\pi_\alpha^\beta(H_\beta)=\pi_\beta(H_0)$ and $\bar{H}_\alpha=\bar{\pi}_\alpha^\beta(\bar{H}_\beta)=\bar{\pi}_\beta(\bar{H})$ are valid the epimorphisms $\pi_\alpha^\beta, \pi_\beta, \bar{\pi}_\alpha^\beta, \bar{\pi}_\beta$ define the continuous surjections ϱ_α^β, ϱ_β, $\bar{\varrho}_\alpha^\beta$, $\bar{\varrho}_\beta$ such that

$$\begin{array}{ccccc}
G_\alpha & \xleftarrow{\pi_\alpha^\beta} & G_\beta & \xleftarrow{\pi_\beta} & G_0 \\
\downarrow{\scriptstyle\chi_\alpha} & & \downarrow{\scriptstyle\chi_\beta} & & \downarrow{\scriptstyle\chi_0} \\
G_\alpha/H_\alpha & \xleftarrow[\varrho_\alpha^\beta]{} & G_\beta/H_\beta & \xleftarrow[\varrho_\beta]{} & G_0/H_0
\end{array}
\qquad
\begin{array}{ccccc}
\bar{G}_\alpha & \xleftarrow{\bar{\pi}_\alpha^\beta} & \bar{G}_\beta & \xleftarrow{\bar{\pi}_\beta} & \bar{G} \\
\downarrow{\scriptstyle\bar{\chi}_\alpha} & & \downarrow{\scriptstyle\bar{\chi}_\beta} & & \downarrow{\scriptstyle\bar{\chi}} \\
\bar{G}_\alpha/\bar{H}_\alpha & \xleftarrow[\bar{\varrho}_\alpha^\beta]{} & \bar{G}_\beta/\bar{H}_\beta & \xleftarrow[\bar{\varrho}_\beta]{} & \bar{G}/\bar{H}
\end{array}$$

Thus inverse systems $\{G_\alpha/H_\alpha,\ \varrho_\alpha^\beta\}$, $\{\bar{G}_\alpha/\bar{H}_\alpha,\ \bar{\varrho}_\alpha^\beta\}$ are obtained which approximate G_0/H_0, $\bar{G}/\bar{H}$ respectively in the sense that the maps

$$\varrho\colon G_0/H_0 \to \varprojlim\{G_\alpha/H_\alpha|\alpha\in\Delta\} \quad \text{and} \quad \bar{\varrho}\colon \bar{G}/\bar{H} \to \varprojlim\{\bar{G}_\alpha/\bar{H}_\alpha|\alpha\in\Delta\}$$

which are defined by $\varrho(gH_0)=\{\varrho_\alpha(gH_0)|\alpha\in\Delta\}$ and $\bar{\varrho}(\bar{g}\bar{H})=\{\bar{\varrho}_\alpha(\bar{g}\bar{H})|\alpha\in\Delta\}$ prove to be homeomorphisms according to standard theorems (see [7], vol. II, pp. 99—122 and [11]). Consider now the following diagram

$$\begin{array}{ccccc}
\bar{G}_\alpha/\bar{H}_\alpha & \xleftarrow{\ \bar{\varrho}_\alpha^\beta\ } & \bar{G}_\beta/\bar{H}_\beta & \xleftarrow{\ \bar{\varrho}_\beta\ } & \bar{G}/\bar{H} \\
\downarrow{\scriptstyle \xi_\alpha} & & \downarrow{\scriptstyle \xi_\beta} & & \downarrow{\scriptstyle \xi} \\
G_\alpha/H_\alpha & \xleftarrow[\ \varrho_\alpha^\beta\ ]{} & G_\beta/H_\beta & \xleftarrow[\ \varrho_\beta\ ]{} & G_0/H_0
\end{array}$$

In fact stipulations above yield that

$$\xi_\beta\circ\bar{\varrho}_\beta(\bar{g}\bar{H}) = \xi_\beta\circ\bar{\varrho}_\beta\circ\bar{\chi}(\bar{g}) = \xi_\beta\circ\bar{\chi}_\beta\circ\bar{\pi}_\beta(\bar{g}) = \chi_\beta\circ\lambda_\beta\circ\bar{\pi}_\beta(\bar{g}),$$

$$\varrho_\beta\circ\xi(\bar{g}\bar{H}) = \varrho_\beta\circ\xi\circ\bar{\chi}(\bar{g}) = \varrho_\beta\circ\chi_0\circ\lambda(\bar{g}) = \chi_\beta\circ\pi_\beta\circ\lambda(\bar{g}) = \chi_\beta\circ\lambda_\beta\circ\bar{\pi}_\beta(\bar{g})$$

and $\xi_\alpha\circ\bar{\varrho}_\alpha^\beta(\bar{g}_\beta\bar{H}_\beta) = \xi_\alpha\circ\bar{\varrho}_\alpha(\bar{g}\bar{H})$, $\quad \varrho_\alpha^\beta\circ\xi_\beta(\bar{g}_\beta\bar{H}_\beta) = \varrho_\alpha^\beta\circ\xi_\beta\circ\bar{\varrho}_\beta(\bar{g}\bar{H}) = \varrho_\alpha\circ\xi(\bar{g}\bar{H})$

which show that the diagram commutes. But the commutativity of this diagram implies that a map

$$\xi_\infty\colon \varprojlim\{\bar{G}_\alpha/\bar{H}_\alpha|\alpha\in\Delta\} \to \varprojlim\{G_\alpha/H_\alpha|\alpha\in\Delta\}$$

is defined by $\xi_\infty(\{\bar{g}_\alpha\bar{H}_\alpha|\alpha\in\Delta\})=\{\xi_\alpha(\bar{g}_\alpha\bar{H}_\alpha)|\alpha\in\Delta\}$. Moreover by the same reason even

$$\xi = \varrho^{-1}\circ\xi_\infty\circ\bar{\varrho}$$

is valid. The map $\bar{\varrho}_\alpha\circ\bar{\varepsilon}$ is covering now the map $\varrho_\alpha\circ\varepsilon_0\circ\eta$ since $\xi_\alpha\circ\bar{\varrho}_\alpha\circ\bar{\varepsilon}=\varrho_\alpha\circ\varepsilon_0\circ\eta$ is valid and $\varrho_\alpha\circ\varkappa_0'$ is a homotopy from $\varrho_\alpha\circ\varepsilon_0\circ\eta$ to a constant map. Since ξ_α is a covering map of manifolds there exists a unique lift $\bar{\varkappa}_\alpha$ of $\varrho_\alpha\circ\varkappa_0'$; in other words there exists a unique continuous map

$$\bar{\varkappa}_\alpha\colon \bar{A}/\bar{A}_H\times I \to \bar{G}_\alpha/\bar{H}_\alpha$$

such that $\xi_\alpha\circ\bar{\varkappa}_\alpha(x,\tau)=\varrho_\alpha\circ\bar{\varkappa}_0'(x,\tau)$ and $\bar{\varkappa}_\alpha(x,0)=\bar{\varrho}_\alpha\circ\bar{\varepsilon}(x)$ for $x\in\bar{A}/\bar{A}_H$, $\tau\in I$. Consider now the continuous map

$$\bar{\varkappa}_\infty\colon \bar{A}/\bar{A}_H\times I \to \Pi\{\bar{G}_\alpha/\bar{H}_\alpha|\alpha\in\Delta\}$$

which is defined by $\bar{\varkappa}_\infty(x,\tau)=\{\bar{\varkappa}_\alpha(x,\tau)|\alpha\in\Delta\}$. The image of this map $\bar{\varkappa}_\infty$ is actually in the subset $\varprojlim\{\bar{G}_\alpha/\bar{H}_\alpha|\alpha\in\Delta\}$ of the cartesian product $\Pi\{\bar{G}_\alpha/\bar{H}_\alpha|\alpha\in\Delta\}$. In order to verify this observe that $\bar{\varrho}_\alpha^\beta\circ\bar{\varkappa}_\beta$ is a homotopy of $\bar{\varrho}_\alpha\circ\bar{\varepsilon}$ and covers $\varrho_\alpha\circ\varkappa_0'$ since $\bar{\varrho}_\alpha^\beta\circ\bar{\varkappa}_\beta(x,0)=\bar{\varrho}_\alpha^\beta\circ\bar{\varrho}_\beta\circ\bar{\varepsilon}(x)=\bar{\varrho}_\alpha\circ\bar{\varepsilon}(x)$ and $\xi_\alpha\circ\bar{\varrho}_\alpha^\beta\circ\bar{\varkappa}_\beta(x,\tau)=\varrho_\alpha^\beta\circ\xi_\beta\circ\bar{\varkappa}_\beta(x,\tau)=\varrho_\alpha^\beta\circ$

$\circ \varrho_\beta \circ \varkappa_0'(x, \tau) = \varrho_\alpha \circ \varkappa_0'(x, \tau)$ for $x \in \bar{A}/\bar{A}_{II}$, $\tau \in I$. Thus by the uniqueness of the lift $\bar{\varkappa}_\alpha = \bar{\varrho}_\alpha^\beta \circ \bar{\varkappa}_\beta$ holds. Now the formerly mentioned lift of the homotopy $\varkappa_0'$ is obviously given by the continuous map

$$\bar{\varkappa} = \bar{\varrho}^{-1} \circ \bar{\varkappa}_\infty : \bar{A}/\bar{A}_{II} \times I \to \bar{G}/\bar{H}$$

which is a homotopy from $\bar{\varepsilon}$ to a constant map. Let $\sigma: (\bar{A}/\bar{A}_{II}) \times (\bar{B}/\bar{B}_{II}) \to \bar{A}/\bar{A}_{II}$ be the canonical projection, then the continuous map

$$\sigma \circ \bar{\varkappa} : \bar{A}/\bar{A}_{II} \times I \to \bar{A}/\bar{A}_{II}$$

defines a contraction of $\bar{A}/\bar{A}_{II}$ over itself. Thus by the above cited result of Borel $\bar{A}_{II} = \bar{A}$ holds. But then $A/H \cap A$ consists of a single point since $\psi \circ \nu \circ \eta$ is surjective. Consequently $A \subset H$ is valid.

The main result of this paper from which the solution of the characterization problem directly follows is given by the following

Theorem 3. *Let G be a locally compact group and H a closed subgroup such that the coset space G/H is locally contractible. Then G/H is a free union of manifolds which are coset spaces of Lie groups.*

Proof. Let $G^* \subset G$ be an open subgroup which can be approximated by Lie groups. Then in consequence of Lemma 5 and Theorem 2 it suffices to show that G^*/H^*, where $H^* = H \cap G^*$, is homeomorphic to the coset space of a Lie group.

Let A be a locally factorizing invariant subgroup of G^*. Then according to Lemma 4 this subgroup A defines a locally trivial fiber structure $\varphi: G^*/H^* \to G'/H'$ such that $G' = G^*/A$, $H' = H^*/B$ are Lie groups and the fibers are homeomorphic to A/B where $B = A \cap H^*$. Consider now the point $\pi^*(H^*) \in G^*/H^*$. According to the local triviality of φ there is a neighborhood Z' of $\varphi \circ \pi^*(H^*)$ in G'/H' and a homeomorphism $\Phi: Z' \times A/B \to \bar{Z}$ such that $\varphi \circ \Phi(z', aB) = z'$ where $\bar{Z} = \varphi^{-1}(Z')$ is a neighborhood of $\pi^*(H^*)$ in G^*/H^*. The assumption that G/H is locally contractible yields now the existence of a neighborhood $W \subset \bar{Z}$ of $\pi^*(H^*)$ such that any subset $X \subset W$ can be contracted over $\bar{Z}$ onto $\pi^*(H^*)$. Considering the construction by which a locally factorizing invariant subgroup was obtained from a given approximating inverse system of Lie groups, it is easy to see that there exists a locally factorizing invariant subgroup $\hat{A} \subset A$ of G^* such that if $\hat{\varphi}: G^*/H^* \to \hat{G}/\hat{H}$ is the fiber structure defined by $\hat{A}$ then the fiber containing $\pi^*(H^*)$ is a subset of W. More precisely the fiber of φ containing $\pi^*(H^*)$ is A/B and the fiber of $\hat{\varphi}$ containing $\pi^*(H^*)$ is $\hat{A}/\hat{B}$ where $\hat{B} = H^* \cap \hat{A}$. But $\hat{A}/\hat{B}$ can be identified with $A' = \pi'(\hat{A})$ where $\pi': A \to A/B$ is the canonical projection. Let now

$$\varkappa: A' \times I \to \bar{Z}$$

be a contraction of A' over $\bar{Z}$ onto $\pi^*(H^*)$ where I is the closed unit interval. Moreover consider the map

$$\varrho: Z' \times A/B \to A/B$$

which is the canonical projection of the cartesian product on the second factor. Then by the map

$$\varrho \circ \Phi^{-1} \circ \varkappa: A' \times I \to A/B$$

obviously a contraction of A' over A/B is obtained. But then Lemma 6 yields that $\hat{A} \subset B$ is valid. This in turn implies that $\hat{B} = \hat{A}$ and consequently G^*/H^* is homeomorphic to $\hat{G}/\hat{H}$. Since $\hat{G}$ is a Lie group and $\hat{H}$ is a closed subgroup the assertion of the theorem follows.

3. The characterization of transitive Lie group actions

The proof of the following theorem which yields the solution of the general characterization problem is achieved now by the standard argument.

Theorem 4. *Let a σ-compact group G with compact G/G_0 be an effective and transitive topological transformation group of a locally compact and locally contractible space X. Then G is a Lie group and X is homeomorphic to a coset space of G.*

Proof. Let H be a stability subgroup of G then H is a closed subgroup of G and X is homeomorphic to the coset space G/H according to a result of PONTRJAGIN (see [7], vol. I, pp. 167—169). Since G/G_0 is compact G can be approximated by Lie groups (see [6], pp. 175—176). Let now A be the locally factorizing invariant subgroup of G given in the proof of the preceding theorem. Then $B = A$ holds and implies that $A \subset H$. But then the assumption that G acts effectively yields that $A = \{e\}$. Consequently G is a Lie group and the assertion follows.

In the special case when $H = \{e\}$ holds Theorem 3 yields the following

Corollary. *A locally compact group with compact G/G_0 is a Lie group if and only if G is locally contractible.*

This topological characterization of Lie groups can be obtained directly from results of K. H. HOFMANN as well ([4], p. 59). In fact local contractibility implies the hypothesis of the Main Lemma there and wipes out the factors Z and T too.

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