On cosine operator functions in Banach spaces

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A cosine operator function C in a complex Banach space X is a mapping of the field of real numbers R into $B(X)$, the space of bounded linear operators on X to X, satisfying $C(0)=I$ (the identical operator) and

$$C(s+t)+C(s-t) = 2C(s)C(t) \tag{1}$$

for $s, t \in R$. Throughout this paper we will also assume that the operator function $C(s)$ is strongly continuous on R (cf. [7]. Def. 3.2.2).

Strongly continuous cosine operator functions in Banach spaces have been considered e.g. in [2], [4], [6], [11], while generalizations in some linear topological spaces ([4], [13]) or special results in Hilbert spaces ([5], [12]) have also been presented. S. KUREPA [8] has given results concerning cosine operator functions continuous on R in the sense of the uniform operator topology.

The aim of this paper is to present further new results about strongly continuous cosine operator functions. The investigations comprise a perturbation theorem, the concept of the adjoint cosine operator function, spectral theorems and Taylor's formula for cosine functions. The basic concept is that of the generator operator A of the cosine operator function C, which is defined as $Ax=\lim\limits_{t\to 0}\frac{2}{t^2}\{C(t)-I\}x$ for exactly those $x\in X$ for which this limit exists in the norm topology of X. It is known that A is a densely defined closed linear operator and $A=C''(0)$ if the derivatives of $C(s)$ are defined in a similar way. The main method of investigation makes use of the fact that for some $w\geqq 0$ every complex z with $\operatorname{Re} z>w$ satisfies $z^2\in\varrho(A)$ (the resolvent set of A) and for every $x\in X$

$$zR(z^2; A)x = \int_0^\infty e^{-zt}C(t)x\,dt \tag{2}$$

where $R(z; A)$ denotes the resolvent operator of A. Thus the method of investigation is akin to that of the semi-groups of operators and therefore the proofs will be given in a concise form.

1.

The following perturbation theorem generalizes [4—I], Lemma 6.1.

Theorem 1. *Let A be the generator of the cosine operator function $C(s; A)$ and B a bounded linear operator in X. Then $A+B$ is the generator of a cosine operator function $S(s; A+B)$ and $\lim_{\|B\|\to 0} \|S(s; A+B)-C(s; A)\|=0$ uniformly on every compact $K\subset R$.*

Proof. Since C is a cosine function, there exist positive numbers M and w such that $s\geqq 0$ implies $\|C(s)\|\leqq Me^{ws}$. Define $T(s)x=\int_0^s C(t)x\,dt$ ($s\geqq 0$, $x\in X$), then $\|T(s)\|\leqq\frac{M}{w}e^{ws}$. Put $f_0(s)=\|C(s)\|$, $f(s)=\|T(s)\|$, $f_n(s)=\|B\|\int_0^s f(s-t)f_{n-1}(t)\,dt$ ($s\geqq 0$; $n=1,2,\dots$), then f_n is Lebesgue integrable on every $[0,a]$, $a>0$. Moreover, define $S_0(s)=C(s)$, $S_n(s)x=\int_0^s T(s-t)BS_{n-1}(t)x\,dt$ ($s\geqq 0$, $x\in X$), then we have for $n=0, 1, 2, \dots$:

1) $S_n(s)x$ is continuous in s for $s>0$, $x\in X$,
2) $\|S_n(s)\|\leqq f_n(s)$ for $s\geqq 0$

as it is seen by induction using [3], VIII. 1. 21.

Introduce the notation $w_0=w+\frac{M}{w}\cdot\|B\|$, then for $p>w_0$

$$\|B\|\int_0^\infty e^{-ps}f(s)\,ds\leqq\frac{M\|B\|}{w(p-w)}=r(p, B)=r<1$$

and induction gives for $n=0, 1, 2, \dots$ that $f_n(s)\leqq Me^{ps}r^n$. Indeed, this is true for $n=0$, and the validity for $n-1$ implies

$$f_n(s)\leqq Me^{ps}r^{n-1}\|B\|\int_0^s e^{-pt}f(t)\,dt\leqq Me^{ps}r^n. \tag{3}$$

From (3) we obtain that $S_n(s)x$ is continuous at $s=0$ from the right, the series $S(s)=\sum_{n=0}^\infty S_n(s)$ converges absolutely for $s\geqq 0$ and $\|S(s)\|\leqq\frac{Me^{ps}}{1-r}$, moreover $S(s)x$ is continuous in s for $s\geqq 0$.

If $\operatorname{Re} v>w_0$, then (3) gives $\int_0^\infty e^{-s\operatorname{Re} v}\|BS_n(s)x\|\,ds<\infty$, and we get for $n=0, 1, 2, \dots$

$$\int_0^\infty e^{-vs}S_n(s)x\,ds=vR(v^2; A)\{BR(v^2; A)\}^n x \quad (x\in X).$$

Indeed, for $n=0$ see [4-I], Lemma 5.6, and the validity for $n-1$ implies by [3], VIII. 1. 22

$$\int_0^\infty e^{-vs} S_n(s)x\,ds = \int_0^\infty e^{-vs} T(s) B \int_0^\infty e^{-vt} S_{n-1}(t)x\,dt\,ds =$$

(4)

$$= v \int_0^\infty e^{-vs} T(s) \{BR(v^2; A)\}^n x\,ds = vR(v^2; A)\{BR(v^2; A)\}^n x.$$

For $v > w_0$ we have $\|BR(v^2; A)\| \leqq \|B\| \int_0^\infty e^{-vs} \|T(s)\|\,ds \leqq r(v, B) < 1$, therefore $\sum_{n=0}^\infty \{BR(v^2; A)\}^n$ converges absolutely, moreover $\int_0^\infty e^{-vs} S(s)x\,ds = \sum_{n=0}^\infty \int_0^\infty e^{-vs} S_n(s)x\,ds$ $(x \in X)$, by [3], III. 6.16. Now if $D(A+B)=D(A)$, then (4) and [7], Theorem 5.10.4 give $v^2 \in \varrho(A+B)$ and

$$\int_0^\infty e^{-vs} S(s)x\,ds = vR(v^2; A+B)x \quad (x \in X),$$

thus [4-I], Lemma 5.8 yields that S is a cosine operator function with generator $A+B$. For $p > w_0$ we have

$$\|S(s; A+B) - C(s; A)\| \leqq \sum_{n=1}^\infty f_n(s) \leqq Me^{ps} \frac{r(p, B)}{1-r(p, B)} \quad (s \geqq 0)$$

and $\lim_{\|B\| \to 0} r(p, B) = 0$ gives the last assertion of Theorem 1 for $s \geqq 0$, while for $s < 0$ it follows from the fact that every cosine operator function is even in s.

Corollary. *Under the conditions of Theorem* 1 *if* $\|C(s; A)\| \leqq Me^{w|s|}$ $(s \in R,$ $w > 0)$ *and* $p > w + \frac{M}{w} \|B\|$, *then there exists an* $N = N(p) > 0$ *such that* $\|S(s; A+B)\| \leqq$ $\leqq Ne^{p|s|}$ *for* $s \in R$.

In the following part of this section the concept of the adjoint cosine operator function will be defined and investigated. To make complicated formulas more readable, we shall write $\langle x^*, x \rangle$ instead of $x^*(x)$ if $x \in X$, $x^* \in X^*$ (the adjoint space of X). It is clear that if $C: R \to B(X)$ is a strongly continuous cosine operator function, then the mapping $C^*: R \to B(X^*)$ defined by $C^*(s) = C(s)^*$ satisfies (1), $C^*(0) = I^*$, $\|C^*(s)\| = \|C(s)\|$ for $s \in R$, and $C^*(s)$ is continuous on R with respect to the w^*-operator topology of $B(X^*)$. However, it may happen that $C^*(s)$ is not a strongly continuous operator function.

The proof of the following lemmas will be only indicated or omitted.

Lemma 1. 1) *If* $x^* \in D(A^*)$, *then for* $s \in R$ *we have* $C^*(s)x^* \in D(A^*)$ *and* $A^* C^*(s)x^* = C^*(s)A^* x^*$. *For every* $x \in X$

$$\langle \{C^*(s) - I^*\} x^*, x \rangle = \int_0^s (s-t) \langle C^*(t) A^* x^*, x \rangle \, dt.$$

2) $x^* \in D(A^*)$ *if and only if there exists* $\text{w}^*\text{-}\lim_{s \to 0} \frac{1}{s^2} \{C^*(s) - I^*\} x^* = \frac{y^*}{2}$, *and then* $A^* x^* = y^*$.

The proof of 1) makes use of [11], 2.13. and 2.14., while that of 2) of [11], 2.11.

Definition 1.
$$X_0^* = \{x^* \in X^* \colon \lim_{s \to 0} C^*(s) x^* = x^*\}.$$

Lemma 2. 1) X_0^* *is a closed linear subspace of* X^*. *For every* $s \in R$ *we have* $C^*(s) X_0^* \subset X_0^*$.

2) $D(A^*) \subset X_0^*$ *and for* $x^* \in D(A^*)$

$$\|\{C^*(s) - I^*\} x^*\| \leqq \frac{s^2}{2} \|A^* x^*\| \sup_{0 \leqq t \leqq |s|} \|C(t)\|.$$

Definition 2. Let $\{C_0^*(s); s \in R\}$ be the restriction of $\{C^*(s); s \in R\}$ to X_0^*, and A_0^* the generator of the strongly continuous cosine operator function $C_0^*(s)$. C_0^* will be called the *cosine operator function adjoint to C.*

Remark. Lemma 2 implies that C_0^* satisfies (1), and Definition 1 and [11], 2.7 that C_0^* is strongly continuous.

In the next lemma and theorem $\bar{H}$ denotes the closure of $H \subset X^*$ in the norm topology of X^*.

Lemma 3. 1) $D(A_0^*) \subset D(A^*)$ *and* $\overline{D(A^*)} = X_0^*$.

2) $D(A_0^*) = \{x^* \in D(A^*) \colon A^* x^* \in X_0^*\}$, *and* $x^* \in D(A_0^*)$ *implies* $A_0^* x^* = A^* x^*$.

In the following theorem we use the definitions of [7], 14.2 and 14.3.

Theorem 2. *If* A *is a cosine generator, then* A *is a* $\odot$*-operator and* $X^\odot = X_0^*$ (cf. [7], Def. 14.2.1). *Moreover,* $A^\odot = A_0^*$ *and for* $s \in R$ *we have* $C(s)^\odot = C_0^*(s)$ (cf. [7], Def. 14.3.1).

Proof. By assumption, A also generates a semi-group of operators of class $H\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, according to [2]. Hence A is a $\odot$-operator, by [7], 14.4. According to definition $X^\odot = \overline{D(A^*)}$, and Lemma 3 gives $X^\odot = X_0^*$. The second part of Lemma 3 and [7], Def. 14.3.1. imply $A^\odot = A_0^*$, finally $C(s)^\odot = C_0^*(s)$ for every $s \in R$, by Lemma 2.

2.

In the investigation of spectral theorems the next lemma will be fundamental.

Lemma 4. *Suppose C is a cosine operator function, A is its generator, $s \in R$ and $a \in K$ (the complex field). Then $S(s; a)x = \int_0^s \operatorname{sh} a(s-t)C(t)x\,dt$ $(x \in X)$ defines a bounded linear operator in X, for which*

$$(5) \qquad AS(s; a)x = a^2 S(s; a)x + a\{C(s) - \operatorname{ch}(as)\}x \quad (x \in X).$$

Proof. Suppose $x \in D(A)$ and $f\colon R \to K$ is twice continuously differentiable. Then, by [4-I], Lemma 5.4,

$$\int_0^s f(t)C(t)Ax\,dt = \int_0^s f(t)C''(t)x\,dt$$

and integrating by parts, we get by [11], 2.16

$$(6) \qquad \int_0^s C(t)\{[f(t)-f(s)]Ax - f''(t)x\}\,dt = f'(0)x - f'(s)C(s)x.$$

For $a=0$ the assertions of the lemma are trivial. For $a \neq 0$ put $f(t) = \frac{1}{a}e^{at}$ and $f(t) = -\frac{1}{a}e^{-at}$ into (6), then we get after some calculation

$$(7) \qquad S(s; a)Ax = a^2 S(s; a)x + a\{C(s) - \operatorname{ch}(as)\}x$$

and [3], III. 6.20 implies (5) for $x \in D(A)$. Now if $x \in X$, $\lim_{n\to\infty} x_n = x$, $\{x_n\} \subset D(A)$, then $\lim_{n\to\infty} S(s; a)x_n = S(s; a)x$ and there exists $\lim_{n\to\infty} AS(s; a)x_n$, for (5) is true on $D(A)$. Then the closedness of A implies (5) for $x \in X$, and the proof is complete.

In the following theorems $P\sigma$, $C\sigma$ and $R\sigma$ denote the point, continuous and residual spectra. We shall refer to the spectral properties P_v $(v=1, 2, 3)$ of a linear operator T from $D(T) \subset X$ to X, whose definition is as follows (cf. [7], Def. 2.16.2):

P_1: T is not one-to-one,
P_2: $R(T)$, the range of T, is not dense in X,
P_3: there is a sequence $\{x_n\} \subset D(T)$ such that $\|x_n\| = 1$ and $\|Tx_n\| \to 0$.

Theorem 3. *If C is a cosine operator function, A is its generator and $s \in R$, then* $\operatorname{ch}\{s\sqrt{\sigma(A)}\} \subset \sigma\{C(s)\}$. *Similar relations hold if we write $P\sigma$ and (for $s \neq 0$) $C\sigma$ and $R\sigma$, respectively, instead of σ on both sides.*

Proof. We may assume, obviously, that $s\neq 0$. If $a\neq 0$ complex, then for $x\in D(A)$ we have, by Lemma 4,

$$\frac{1}{a}\int_0^s \operatorname{sh} a(s-t)C(t)\{a^2-A\}x\,dt = \{\operatorname{ch}(as)-C(s)\}x. \tag{8}$$

On the other hand, [11], 2.15 gives for $x\in D(A)$

$$\int_0^s (s-t)C(t)\{0-A\}x\,dt = \{\operatorname{ch}0-C(s)\}x. \tag{9}$$

Suppose now $a^2\in\sigma(A)$. If $a\neq 0$, then (8), while if $a=0$, then (9) immediately yield that if a^2-A has the spectral property P_v $(v=1, 2, 3)$, then so does $\operatorname{ch}(as)-C(s)$. This gives the statements of the theorem.

The converse relation for the point spectra is given in the following

Theorem 4. *If $s\in R$, $s\neq 0$, $p\in P\sigma\{C(s)\}$ and $\{r_n\}$ is the set of all complex solutions of the equation* $\operatorname{ch}(rs)=p$, *then $r_n^2\in P\sigma(A)$ for some n. Therefore,* $\operatorname{ch}\{s\sqrt{P\sigma(A)}\}= =P\sigma\{C(s)\}$.

Proof. If $a\neq 0$ complex and $\operatorname{ch}(as)\in\varrho\{C(s)\}$, then $R(\operatorname{ch}(as); C(s))$ commutes with $S(s; a)$ and $a^2\in\varrho(A)$, by Theorem 3. Moreover, by Lemma 4,

$$R(a^2; A) = \frac{1}{a}S(s; a)R(\operatorname{ch}(as); C(s)). \tag{10}$$

Suppose $p\in P\sigma\{C(s)\}$, $s\neq 0$, $M=\{x\in X: C(s)x=px\}$. Then M is a nontrivial closed linear subspace of X, invariant for $C(t)$, $t\in R$. In the remainder of this proof $C(t)$ $(t\in R)$ and A denote the restrictions of these operators to M, unless explicitly stated otherwise. Thus if $\operatorname{ch}(as)\neq p$, then $\operatorname{ch}(as)\in\varrho\{C(s)\}$ and, by (10),

$$R(a^2; A) = \frac{1}{a}(\operatorname{ch}(as)-p)^{-1}S(s; a) \quad (\text{if } a\neq 0). \tag{11}$$

If for some complex r_n for which $\operatorname{ch}(r_n s)=p$, $S(s; r_n)$ is not the zero operator in M, then the resolvent $R(v; a)$ has a pole at $v=r_n^2$, by (11), consequently $r_n^2\in P\sigma(A)$ even if A is considered on all of $D(A)$, thus the theorem is true. Therefore we assume that $S(s; r_n)=0$ on M for every r_n for which $\operatorname{ch}(r_n s)=p$.

Put $\{r_n\}=\{a_n\}\cup\{b_n\}$ where $a_n=a_0+i\frac{\pi}{s}2n$, $b_n=-a_0+i\frac{\pi}{s}2n$ (n integer) are all solutions of the above equation. By our assumption, we obtain for $x\in M$

$$\int_0^s C(s-t)\,\mathrm{sh}\,(a_0t)\cos\left(\frac{\pi}{s}2nt\right)\cdot x\,dt = \tag{12}$$
$$= \int_0^s C(s-t)\,\mathrm{ch}\,(a_0t)\sin\left(\frac{\pi}{s}2nt\right)x\,dt = 0.$$

$C(s)$ is an even function, therefore we may assume $s>0$. Fix $x\in M$, and define the functions $f, g\colon R\setminus\{ns;\,n \text{ integer}\}\to X$ to be periodic with period s, and for $t\in(0,s)$

$$f(t) = C(s-t)\,\mathrm{ch}\,(a_0t)x,\quad g(t) = C(s-t)\,\mathrm{sh}\,(a_0t)x. \tag{13}$$

Then the sine Fourier coefficients of f and the cosine coefficients of g vanish by (12), and their Fourier series are $(C, 1)$-summable to $f(t)$ and $g(t)$, respectively, for $t\in(0,s)+ns$ (n integer) as in the numerical-valued case. Hence f is even and g is odd, and we obtain for $t\in(0,s)$ that on M

$$C(t)e^{a_0s} = C(s-t) = C(t)e^{-a_0s}. \tag{14}$$

Since M is a nontrivial subspace, thus we can not have $C(t)M=\{0\}$ for $t\in(0,s)$, hence $e^{a_0s}=\pm1$ and $p=\mathrm{ch}\,(a_0s)=1$ or else $p=-1$.

Now if $e^{a_0s}=-1$, then by (14) $C\left(\frac{s}{2}\right)=0$ on M and $C(t+s)=-C(t)$ for $t\in R$. It can be shown that $E(t)=C(t)+iC\left(t+\frac{s}{2}\right)$ is a strongly continuous group of operators for which $E(s)=-I$ and whose generator G satisfies $G^2=A$ (cf. [9]). But then $-1\in P\sigma\{E(s)\}$ and [7], Theorem 16.7.2 give that for some complex r, for which $\mathrm{ch}\,(rs)=-1$, $r\in P\sigma(G)$ and, consequently, $r^2\in P\sigma(A)$ holds even if A is considered on all of $D(A)$.

Finally, if $e^{a_0s}=1$, then using (14) it can be shown that, on M, $C(t)$ is periodic with period s. According to [6], $P\sigma(A)=\sigma(A)$ is then nonvoid and $P\sigma(A)\subset\{r_n^2\}$, thus the proof is complete.

In view of our results concerning the adjoint cosine operator function and the point spectra, in the following two theorems a similar reasoning can be applied as in [7], Theorem 16.7.3 and 16.7.4.

Theorem 5. *If $p\in R\sigma\{C(s)\}$ and $\{r_n\}$ is the set of all complex solutions of the equation* ch $(rs)=p$, *then $r_n^2\in R\sigma(A)$ for some n, and $r_n^2\notin P\sigma(A)$ for every n. Moreover we have $p\in P\sigma\{C_0^*(s)\}$.*

Proof. We only remark that, by Theorem 2, A is a $\odot$-operator and for $t\in R$, $C(t)$ commutes with A in the sense of [7], Def. 14.3.2, for there is a $w\geqq 0$ such that $\operatorname{Re} v > w$ implies

$$R(v^2;A)C(t)x = \frac{1}{v}\int_0^\infty e^{-vu}C(u)C(t)x\,du = C(t)R(v^2;A)x \quad (x\in X).$$

Now the proof is similar to that of [7], Theorem 16.7.3.

Theorem 6. *If $p\in C\sigma\{C(s)\}$ and r_n as in Theorem 5, then $\{r_n^2\}\subset C\sigma(A)\cup\varrho(A)$. It can happen that every $r_n^2\in\varrho(A)$.*

Proof. The first assertion follows from Theorem 3, and the following example proves the second one. Let X be the complex l_2 space, and for $\{z_n; n=1, 2, \ldots\}\in l_2$, $s\in R$ put $C(s)\{z_n\}=\{\cos(ns)z_n\}$. Then $A\{z_n\}=\{-n^2z_n\}$ with $D(A)=\left\{\{z_n\}\in l_2;\ \sum_{n=1}^{\infty} n^4|z_n|^2<\infty\right\}$, and $\sigma(A)=P\sigma(A)=\{-n^2;\ n=1,2,\ldots\}$. Clearly, $P\sigma\{C(1)\}=\{\cos n;\ n=1,2,\ldots\}$, $K\setminus[-1,1]\subset\varrho\{C(1)\}$ and Theorem 5 implies $C\sigma\{C(1)\}=[-1,1]\setminus\{\cos n; n=1,2,\ldots\}$. Thus the second assertion is also proved.

The next theorem (Taylor's formula for cosine operator functions) generalizes [11], 2.15.

Theorem 7. *Suppose C is a cosine operator function, A is its generator and $x\in D(A^n)$ (n positive integer). Then for $t\in R$*

$$C(t)x = x+\frac{t^2}{2!}Ax+\cdots+\frac{t^{2n-2}}{(2n-2)!}A^{n-1}x+\int_0^t\frac{(t-s)^{2n-1}}{(2n-1)!}C(s)A^nx\,ds. \tag{15}$$

Proof. It is known that, for $x\in D(A)$, $C(t)x$ is twice continuously differentiable on R, $C(t)x\in D(A)$, and $C''(t)x=C(t)Ax=AC(t)x$ for every $t\in R$; moreover, $C'(0)x=0$, cf. [4-I], [11]. From these facts it can be deduced by induction that for $x\in D(A^n)$ $C(t)x$ is $2n$ times continuously differentiable on R, $C^{(2n)}(t)x=C(t)A^nx=$ $=A^nC(t)x$ for $t\in R$, and $C^{(2k-1)}(0)x=0$ whenever $1\leqq k\leqq n$. Hence the Taylor theorem for vector-valued functions (see e.g. [10], (IV, 9; 47)) gives the assertion of the theorem.

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(Received June 12, 1973, revised December 29, 1973)