Representations of groups by automorphisms of objects in a category

By T. MATOLCSI in Szeged

1. Introduction

The theory of group representations has always developed in close connection with physics. Group representation meant first a group homomorphism into the group of invertible linear transformations of a linear vector space. Then it was recognized that in quantum mechanics unitary ray representations play the fundamental role rather than usual representations; here group elements are represented by certain equivalence classes of unitary or antiunitary operators on a Hilbert space. Note that ray representations do not differ very much from, and can be reduced to, usual representations ([1]).

Recently, however, in the axiomatic foundation of quantum mechanics (or, rather, of general mechanics) it turned out that one need to represent groups in a so far unusual sense. Here one defines representations as group homomorphisms into the group of automorphisms of various algebraic and topological structures ([2], [3]). It has become clear that unitary ray representations are nothing else than representations by automorphisms of the lattice of closed linear subspaces of a separable Hilbert space ([2]). Usual representations can be formulated as representations by automorphisms of a linear vector space. Topological transformation groups can be considered as a special sort of representations by automorphisms of a topological space.

This suggests how we should define the standard notions of group representations in the most general form using the theory of categories. Then, first of all, the question arises, how we can state the generalization of the celebrated Schur lemma. Schur's lemma has several different formulations in the literature. For convenience we cite the most important ones. All linear vector spaces are over the field $\mathbf{C}$ of complex numbers and G is a given group in the sequel.

1) Let $A^{(i)}$ be irreducible representations of G on the finite dimensional linear vector spaces $V^{(i)}$ $(i=1, 2)$. If for a linear map $T: V^{(1)} \to V^{(2)}$ we have $TA_g^{(1)} = A_g^{(2)} T$ for all $g \in G$, then either $T=0$ or T is one-one and onto.

2) Let $U^{(i)}$ be irreducible unitary representations of G on the Hilbert spaces $H^{(i)}$ $(i=1,2)$. If for a bounded linear map $T:H^{(1)}\to H^{(2)}$ we have $TU_g^{(1)}=U_g^{(2)}T$ for all $g\in G$, then $T=\lambda W$ where $\lambda\in\mathbf{C}$ and W is a unitary map.

3) Let U be an irreducible unitary representation of G. If for a bounded linear operator we have $TU_g=U_gT$ for all $g\in G$, then $T=\lambda I$. (I is the identity operator.)

4) Let A be an irreducible representation of G on the linear vector space V. If for a linear map $T:V\to V$ we have $TA_g=A_gT$ for all $g\in G$, then $T=\lambda I$.

We shall refer to the versions 1), 2) and 3), 4) as the first and the second type of Schur's lemma, respectively. At first sight one would say that the second type is a more or less immediate consequence of the first one. In reality, however, in the case of unitary representations both types can be considered as a consequence of each other ([4]). We are going to find a general framework in which the nature of the different types of Schur's lemma becomes more apparent.

2. Basic notions

We use, for our purposes, the language and some results of the theory of categories, which may be found, for instance, in [5], [6], [7].

The notion of subobjects will have a crucial importance for us. Let $\mathscr{C}$ denote a given category. In general, a pair (U, u) is called a subobject of $X\in\operatorname{Ob}\mathscr{C}$ if $U\in\operatorname{Ob}\mathscr{C}$, $u\in\operatorname{Mor}(U, X)$ and u is monic. Let (U, u) and (V, v) be subobjects of X; (U, u) majorates (or is greater than) (V, v) if there is a $w\in\operatorname{Mor}(V, U)$ so that $uw=v$. If also (V, v) majorates (U, u) then we say that the two subobjects are equivalent; in this case w is an isomorphism. Equivalent subobjects are considered to be the same.

In a variety of applications this definition of subobjects is not suitable because its content is too large. For instance, in the category of topological spaces and continuous maps a subset of a topological space equipped with a topology finer than the induced topology would be a subspace. That is why we make another definition. We require that u have some property p and we say that u is a p-morphism and (U, u) is a p-subobject. In concrete categories — in which objects are sets with some structure and morphisms are certain maps — there is, generally, a natural way to choose the property p. For instance, in the category mentioned above a monomorphism u is a p-morphism if it cannot be factored in the form $u=vw$ where v is a monomorphism, w is a bimorphism but is not an isomorphism. Of course, in order that the definition of p-subobjects be consistent, the following conditions must be fulfilled:

1) isomorphisms are p-morphisms;
2) the composition of two p-morphisms is a p-morphism;
3) if u and uv are p-morphisms then v is a p-morphism.

If there is no need to mention explicitly the object U or the monomorphism u, we use also the notation (u) or (U) for the subobject (U, u). For example (X) and (id_X) denote the trivial subobject (X, id_X).

Now we can turn to our aim.

Definition 1. A subobject (U, u) of $X \in \mathrm{Ob}\,\mathscr{C})$ is *invariant* for $a \in \mathrm{End}\,(X)$ if there is a $b \in \mathrm{End}\,(U)$ such that $au = ub$.

It is routine to check that a subobject equivalent to (U, u) is also invariant for a; so the definition is consistent. In concrete categories Definition 1 coincides with the usual definition of invariant subspaces, subalgebras etc. Note, lastly, that b is uniquely determined because u is a monomorphism.

Now we give an easy but important assertion concerning invariant subobjects. A subobject (U, u) of X will be called initial if $\mathrm{Mor}\,(U, X) = \{u\}$. A zero object, for instance, is an initial subobject of all objects.

Proposition 1. *(X) and all initial subobjects of X are invariant for all automorphisms of X.*

Let us given now a group G and let $\mathscr{G}$ be the category whose only object is G and whose morphisms are the elements of G with group multiplication as composition of morphisms.

Let us construct the category $\mathscr{C}^{\mathscr{G}}$ whose objects are covariant functors from $\mathscr{G}$ into $\mathscr{C}$ and whose morphisms are the natural transformations (functorial morphisms) between such functors. The category ${}^{\mathscr{G}}\mathscr{C}$ of contravariant functors is constructed similarly. A functor $A: \mathscr{G} \to \mathscr{C}$ associates an object $A(G)$ of $\mathscr{C}$ with G and an automorphism A_g of $A(G)$ with each $g \in G$. A natural transformation between the functors A and B is now a morphism $f: A(G) \to B(G)$ in $\mathscr{C}$ such that $fA_g = B_g f$ for all $g \in G$. We shall use the notation $\{A_G = A_g : g \in G\}$.

Definition 2. An object A of $\mathscr{C}^{\mathscr{G}}$ resp. of ${}^{\mathscr{G}}\mathscr{C}$ is called a *left* resp. *right representation* of G in $\mathscr{C}$. *Faithful* representations are faithful functors. A representation A is called p-*irreducible* if there is no p-subobject of $A(G)$ invariant for all A_g and not invariant for all automorphisms of $A(G)$. Two representations A and B are said to be *equivalent* if there is a natural equivalence (functorial isomorphism) between A and B. A morphism $A \to B$ in $\mathscr{C}^{\mathscr{G}}$ or in ${}^{\mathscr{G}}\mathscr{C}$ is called a *G-intertwiner* from A into B.

In view of physical applications we introduce another sort of irreducibility.

Definition 3. Let $\mathscr{C}$ be a concrete category. A representation A of the group G in $\mathscr{C}$ is *weakly irreducible* if there is no $x \in A(G)$ invariant for all A_g and not invariant for all automorphisms of $A(G)$.

If $x \in A(G)$ is invariant for an automorphism of $A(G)$, then, in many important cases, the subobject generated by the element x is also invariant for the automorphism. If so, irreducibility implies weak irreducibility.

Lastly, before going further, we introduce three categories which are fundamental in the theory of usual representations. Let *Vect* be the category of complex linear vector spaces and linear maps. *Vectf* is its full subcategory whose objects are finite dimensional vector spaces. Finally let *Hil* be the category of Hilbert spaces and linear contractions. The p-subobjects are chosen as usually in the theory of such spaces.

3. General results

Let us see now in general, how the irreducible representations can be characterized in a similar fashion as the Schur lemma does. Let us start with the second type. It is based on the relation between the commutant of an endomorphism and subobjects invariant for a representation. For this reason, first, we introduce the following notations.

Let X be an arbitrary object of $\mathscr{C}$ and let $E \subset \mathrm{End}(X)$. Then we define

$E' := \{b \in \mathrm{End}(X) : ab = ba$ for all $a \in E\}$;
$E^+ := E' \cap \mathrm{Aut}(X)$;
$E^{\mathrm{p}} :=$ the class of p-subobjects invariant for all $a \in E$.

We find that if $E \subset F \subset \mathrm{End}(X)$ then $F' \subset E'$ and $F^{\mathrm{p}} \subset E^{\mathrm{p}}$. A p-irreducible representation A of G on X is characterized by $A_G^{\mathrm{p}} = \mathrm{Aut}(X)^{\mathrm{p}}$. For being able to say more we impose a condition on X which makes sharper the relation $E^+ \subset \mathrm{Aut}(X)$, $\mathrm{Aut}(X)^{\mathrm{p}} \subset E^{+\mathrm{p}}$. We formulate it as

Condition 1. Let $a \in \mathrm{End}(X)$. If $\{a\}^+ \neq \mathrm{Aut}(X)$ then $\mathrm{Aut}(X)^{\mathrm{p}} \neq \{a\}^{+\mathrm{p}}$.

Theorem 1. *Let A be a* p-*irreducible representation of the group G in $\mathscr{C}$ and suppose $X = A(G)$ satisfies* Condition 1. *If $a \in \mathrm{End}(X)$ is a G-intertwiner then* $\{a\}^+ = \mathrm{Aut}(X)$.

Proof. The assertion of the theorem can be formulated so that if $A_G \subset \{a\}^+$ then $\{a\}^+ = \mathrm{Aut}(X)$. Since $A_G \subset \{a\}^+ \subset \mathrm{Aut}(X)$, we have $\mathrm{Aut}(X)^{\mathrm{p}} \subset \{a\}^{+\mathrm{p}} \subset A_G^{\mathrm{p}}$, but from the irreducibility of A it follows that $A_G^{\mathrm{p}} = \mathrm{Aut}(X)^{\mathrm{p}}$. Thus by Condition 1 we conclude that $\{a\}^+ = \mathrm{Aut}(X)$.

Let us consider, as examples, the categories *Vectf* and *Hil*. Objects of both categories satisfy Condition 1 and we have from Theorem 1 the versions of the second type of Schur's lemma. Indeed, because of the fact that if $a \neq 0$ is a bounded linear operator on a Hilbert space then $a/\|a\|$ is a contraction, Condition 1 says in both cases that if $a \neq \lambda\,\mathrm{id}$ $(\lambda \in \mathbf{C})$ is a bounded linear operator, then there exists a non trivial closed linear subspace invariant for all automorphisms commuting with a; in *Vectf* such invariant subspaces are eigenspaces of a, in *Hil* they are the subspaces corresponding to the spectral families of the self-adjoint operators $a+a^*$ and $i(a-a^*)$.

Two simple examples show that Condition 1 does not always hold but it does for an object different from the previous ones. First take the category of sets and maps where p-subobjects are subsets. Here $\{a\}^+ \neq \mathrm{Aut}(X)$ for all $a \in \mathrm{End}(X)$ and $\mathrm{Aut}(X)^{\mathrm{p}} = \{(X), (\emptyset)\}$. Let X be a finite set; if a is a cyclic permutation of elements then $\{a\}^+ = \{(X), (\emptyset)\}$ and Condition 1 fails for X. Secondly consider the category of partially ordered sets and monotone maps, where p-subobjects are subsets with induced ordering. Let $X = \{0, x, y, 1\}$ where x and y are not related. The only automorphism of X, besides the identity, is the one-to-one monotone map b defined by $b(x) = y$. Thus if $\{a\}^+ \neq \mathrm{Aut}(X)$ then $\{a\}^+ = \{\mathrm{id}_X\}$. One can see that $\{\mathrm{id}_X\}^{\mathrm{p}} \neq \mathrm{Aut}(X)^{\mathrm{p}}$, hence Condition 1 is fulfilled.

Let us go further. The first type of Schur's lemma — in the case of linear representations — is based on the relation between subspaces associated with linear maps and subspaces invariant for representations. For this reason we shall be interested in special categories where the corresponding notions — kernels and images — are well defined. There is a sort of categories known in the theory which offers itself for investigations. Unfortunately there is no unique nomenclature in the literature; we shall call a category $\mathscr{C}$ *pre-Abelian* if

1) there is a zero object in $\mathscr{C}$;

2) for all pairs of objects X and Y there is given a commutative group structure on $\mathrm{Mor}(X, Y)$ which is distributive with the composition of morphisms;

3) all morphisms have a kernel and a cokernel.

$\mathscr{C}$ will be in the sequel a pre-Abelian category with zero object N.

The kernel and the image of $f \in \mathrm{Mor}(X, Y)$ are subobjects of X and Y respectively; we denote them by $(\mathrm{Ker} f, \ker f)$ and $(\mathrm{Im} f, \mathrm{im} f)$. Remind that $\mathrm{im} f = \ker(\mathrm{coker} f)$.

Proposition 2. *Let X and Y be objects of a pre-Abelian category. Let $a \in \mathrm{End}(X)$, $b \in \mathrm{End}(Y)$ and $f \in \mathrm{Mor}(X, Y)$ such that $fa = bf$. Then* $(\ker f)$ *is invariant for a and* $(\mathrm{im} f)$ *is invariant for b.*

Proof. The proof of the two assertions are similar, hence we omit the simpler one. Since $\mathrm{coker} f \circ b \circ f = \mathrm{coker} f \circ f \circ a = 0$, there is a u such that $\mathrm{coker} f \circ b = u \circ \mathrm{coker} f$; now it follows that $\mathrm{coker} f \circ b \circ \mathrm{im} f = 0$ and consequently there is a $v \in \mathrm{End}(\mathrm{Im} f)$ with which $b \circ \mathrm{im} f = \mathrm{im} f \circ v$.

It is a natural requirement that in a pre-Abelian category the property p be chosen in such a manner that all kernels (and cosequently all images), as the most important subobjects, be p-subobjects. Doing so we have the next immediate result for group representations.

Theorem 2. *Let G be a group, A and B its representations in a pre-Abelian category. Suppose f is a G-intertwiner from A into B. If A is* p*-irreducible then* $(\ker f)\in \in \mathrm{Aut}\,(A(G))^{\mathrm{p}}$. *If B is* p*-irreducible then* $(\mathrm{im}\, f)\in \mathrm{Aut}\,(B(G))^{\mathrm{p}}$.

This theorem is a generalization of the first type of Schur's lemma, though it has a form rather different from the usual one. We can get it in a more familiar form, imposing a condition on the objects in question.

Condition 2. $\mathrm{Aut}\,(X)^{\mathrm{p}} = \{(N), (X)\}$.

Theorem 3. *Let A and B be representations of the group G in a pre-Abelian category and suppose $A(G)$ and $B(G)$ satisfy* Condition 2. *Let f be a G-intertwiner from A into B. If A is* p*-irreducible then either $f=0$ or f is a monomorphism. If B is* p*-irreducible then either $f=0$ or f is an epimorphism. As a consequence if both A and B are* p*-irreducible then either $f=0$ or f is a bimorphism.*

Proof. In a pre-Abelian category we have the following easily provable relations for a morphism f ([5], [6]):

$$\ker f = 0 \quad \text{if and only if} \quad f \text{ is monic},$$

$$\ker f = \mathrm{id} \quad \text{if and only if} \quad f = 0,$$

$$\mathrm{im}\, f = 0 \quad \text{if and only if} \quad f = 0,$$

$$\mathrm{im}\, f = \mathrm{id} \quad \text{if and only if} \quad f \text{ is epic}.$$

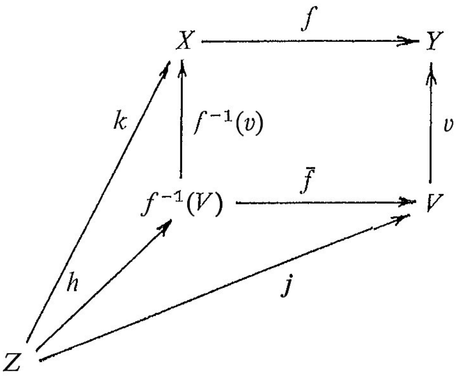
Objects of the categories *Vect* and *Hil* satisfy Condition 2. In *Vect* every bimorphism is an isomorphism, so Theorem 3 gives at once the known version of the first type of Schur's lemma. In *Hil,* as we could expect, the known version is stronger than the one arising from Theorem 3.

There are well-known pre-Abelian categories, for which, consequently, Theorem 2 is valid. Condition 2, however, does not hold in general, but only for certain objects of them. Nevertheless, Theorem 2 is interesting in itself and in the case of the category of Abelian groups, for instance, there are sufficient results concerning characteristic subgroups (invariant for all automorphisms) ([8]) to get further information about homomorphisms intertwining two representations.

On the other hand, there are important categories which are not pre-Abelian; for example, the category of orthocomplemented lattices defined on the base of [3]. Pre-Abelian categories were useful to illuminate the way we should follow. Now we want only that certain images and counterimages (see [7]) exist in the category $\mathscr{C}$.

The image of $f\in \mathrm{Mor}\,(X, Y)$ is the smallest subobject of Y through which f is factored. In other words $f=\mathrm{im}\, f\circ \bar{f}$ and if $f=vk$, where v is a monomorphism, then there exists a morphism h such that $\operatorname{im} f = vh$. Let (U, u) be a subobject of X; the image of fu is called the image of (U, u) under f and is denoted sometimes by $(f(U), f(u))$.

The counterimage of a subobject (V, v) of Y under $f \in \operatorname{Mor}(X, Y)$ is a subobject of X, denoted by $(f^{-1}(V), f^{-1}(v))$, for which there is a morphism $\bar{f}$ such that $ff^{-1}(v) = v\bar{f}$ and if $fk = vj$, there exists a unique morphism h such that the diagram below is commutative:



Proposition 3. *Let X and Y be objects of $\mathscr{C}$. Let $a \in \operatorname{End}(X)$, $b \in \operatorname{Aut}(Y)$ and $f \in \operatorname{Mor}(X, Y)$ such that $fa = bf$. If (U, u) is an invariant subobject for a then $(f(U))$, if exists, is invariant for b. If (V, v) is an invariant subobject for b then $(f^{-1}(V))$, if exists, is invariant for a.*

Proof. Let $au = ua$. Then $bfu = fau = fua$ and we see that it suffices to consider the case $u = \mathrm{id}_X$. We have the factorization $bf = \operatorname{im}(bf) \circ j$ and $f = b^{-1} \circ \operatorname{im}(bf) \circ j$. Since $b^{-1} \circ \operatorname{im}(bf)$ is monic, there is a morphism h so that $b^{-1} \circ \operatorname{im}(bf) \circ h = \operatorname{im} f$, that is $\operatorname{im}(bf) \circ h = b \circ \operatorname{im} f$. Furthermore $\operatorname{im}(bf) = \operatorname{im}(fa)$; now observe that $\operatorname{im}(fa)$ is factored through $\operatorname{im} f$: $\operatorname{im}(fa) = \operatorname{im} f \circ k$ and consequently $b \circ \operatorname{im} f = \operatorname{im} f \circ h \circ k$.

Let $bv = vb$. Then $faf^{-1}(v) = bff^{-1}(v) = bv\bar{f} = vb\bar{f}$. As a consequence there is a morphism h with which $af^{-1}(v) = f^{-1}(v)h$.

Now again we have an immediate result for representations.

Theorem 4. *Let G be a group, A and B its representations in the category $\mathscr{C}$ and let f be a G-intertwiner from A into B. Assume images and counterimages of* p*-subobjects in $\mathscr{C}$ under f exist and are* p*-subobjects. If A is* p*-irreducible then for all $(V) \in \operatorname{Aut}(B(G))^{\mathrm{p}}$ we have $(f^{-1}(V)) \in \operatorname{Aut}(A(G))^{\mathrm{p}}$. If B is* p*-irreducible then for all $(U) \in \operatorname{Aut}(A(G))^{\mathrm{p}}$ we have $(f(U)) \in \operatorname{Aut}(B(G))^{\mathrm{p}}$.*

Now of course, we cannot expect in general a result like Theorem 3, and we do not need it either. Theorem 2 and Theorem 4 are the real generalizations of the first type of Schur's lemma.

Let us see some examples using the notations $X=A(G)$, $Y=B(G)$. In the category *Vectf* Theorem 4 gives the known version. In the category of orthocomplemented lattices, if $\mathrm{Aut}\,(X)^{\mathrm{p}}=\{(M),(X)\}$ where $M=\{0, 1\}$, and the same is true for Y, we obtain the corresponding part of Theorem 3.2 in [3] (weak irreducibility there corresponds to irreducibility here). In the category of partially ordered sets with maximal and minimal elements, if $\mathrm{Aut}\,(X)^{\mathrm{p}}=\{\{0\},\{1\}, M, X\}$ and if the same holds for Y, we have that a monotone map f intertwining two irreducible representations is either trivial ($f(X)=0$ or $f(X)=1$) or $f(0)=0$ and $f(1)=1$; furthermore f is either surjective or empty. Thus if the cardinality of Y is higher than that of X, there is no map $X\to Y$ intertwining irreducible representations.

4. Remarks

In the case of unitary representations the two types of Schur's lemma coincide. Now we see that in the case of linear representations the two types are fully different: we have got a proof of the second one independent of the first one. Of course, one can take $B=A$ in Theorems 2 and 4 to have a result for a morphism commuting with an irreducible representation. If Condition 1 does not hold it is really a result, but with Condition 1 it is implied by Theorem 1. Surely it can happen that by the aid of Theorems 2 and 4 one needs a condition weaker than Condition 1 to have the result of Theorem 1. In this respect the second type can be a corollary of the first one. For example, in the case of an object satisfying Condition 2 in a pre-Abelian category, we should test Condition 1 only for bimorphisms. As another example, let us consider the category of orthocomplemented lattices.

From Theorem 3.2 in [3] it follows that we need Condition 1 only for automorphisms. From Axiom 2 in [3] we conclude that $\{h\}^{\mathrm{p}}\neq\{(M),(X)\}$ for all $h\in\mathrm{Aut}\,(X)$ and we obtain the second type of Schur's lemma (Theorem 3.9 in [3]) for orthocomplemented lattices with $\mathrm{Aut}\,(X)^{\mathrm{p}}=\{(M),(X)\}$. Now we call attention that it is not right here to define irreducibility in general by $A_G^{\mathrm{p}}=\{(M),(X)\}$ as it is done in [3], because there are orthocomplemented lattices for which $\mathrm{Aut}\,(X)^{\mathrm{p}}\neq\{(M),(X)\}$. The σ-algebra of Borel sets in the real line serves as an example: the subalgebra of sets containing denumerably many points or having such a complement is invariant for all automorphisms.*)

Lastly we mention that there are certain other formulations of the Schur lemma, different from the ones given at the beginning of this paper. In a version for unitary representations the intertwining operator need not be bounded but only closed ([4]). Such a result, of course, cannot be reached by the method of categories.

*) This example was given me by my colleague J. Szűcs.

References

[1] V. BARGMANN, On unitary ray representations of groups, *Ann. Math.*, 59 (1954), 1—46.
[2] V. S. VARADARAJAN, *Geometry of Quantum Theory,* Vol. I. Van Nostrand (Princeton, New Jersey, 1968).
[3] S. P. GUDDER, Representations of groups as automorphisms on orthomodular lattices and posets, *Canadian J. Math.*, 23 (1971), 659—673.
[4] M. A. Наймарк, *Нормированные кольца.* Наука (Москва, 1968).
[5] I. BUCUR and A. DELEANU, *Introduction to the Theory of Categories and Functors,* J. Wiley (New York, Sydney, 1968).
[6] H. SCHUBERT, *Categories,* Springer (Berlin, Heidelberg, New York, 1972).
[7] B. PAREIGIS, *Categories and Functors,* Academic Press (New York, London, 1970).
[8] L. FUCHS, *Abelian Groups,* Akadémiai Kiadó (Budapest, 1958).

(Received September 1, 1973)