Operators of class $C_0(N)$ and transitive algebras

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The recent remarkable result of V. I. LOMONOSOV [4], that if an operator (bounded linear transformation) T on a Banach space $\mathfrak{X}$ has a nonzero compact operator in its commutant then T has a nontrivial invariant subspace, has a beautiful and astonishingly simple proof. The proof establishes even stronger results than that stated. Lomonosov does mention one of these in a note at the end of his paper. Another and closely related result is that if $\mathscr{A}$ is a transitive algebra in the Banach algebra $\mathscr{B}(\mathfrak{H})$ of all operators on a separable complex Hilbert space $\mathfrak{H}$ which contains a nonzero compact operator, then $\mathscr{A}$ is weakly dense in $\mathscr{B}(\mathfrak{H})$; see [6].

By a transitive algebra $\mathscr{A}$ we mean a subalgebra of $\mathscr{B}(\mathfrak{H})$ for which there does not exist a nontrivial subspace which is invariant under each operator in $\mathscr{A}$. We should mention that a primary motivation for the study of transitive algebras is that if the only weakly closed transitive algebra is $\mathscr{B}(\mathfrak{H})$, then the invariant subspace conjecture is true, i.e. every operator on a separable complex Hilbert space has a nontrivial invariant subspace. For an excellent discussion of transitive algebras and the history of their development see the monograph by RADJAVI and ROSENTHAL [6; particularly Chapter 8 and 10].

In this paper, we establish that if T is a contraction on $\mathfrak{H}$ such that T^n and T^{*n} go strongly to zero as $n \to \infty$, and if the ranks of $I-T^*T$ and $I-TT^*$ are finite and equal (if N is this rank, then T is said to be of class $C_0(N)$, see [10; p. 350]; also finiteness implies their equality [10; Theorem VI.5.2]), then any transitive algebra that contains T is weakly dense in $\mathscr{B}(\mathfrak{H})$.

The essential underlying result for our study is that if T is in $C_0(N)$ then T commutes with a particularly simple nonzero compact operator, and this is established by working within the functional model $\mathbf{T}$ for T (see [8] or [10]) where the structure of commuting compacts is well understood (see [7] for $N=1$; [5] for $N \geqq 1$). Finally, the result is reached by using the transitive algebra result which followed from

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Lomonosov's proof and noting that the specific nature of this commuting compact implies that it is in the weakly closed algebra $\mathscr{A}_{\mathbf{T}}$ generated by $\mathbf{I}$ and $\mathbf{T}$.

The functional model $\mathbf{T}$ of T in $C_0(N)$ on the space $\mathbf{H}$ is defined by

$$\mathbf{H} = H^2(\mathfrak{E})\ominus \Theta H^2(\mathfrak{E}) \quad \text{and} \quad (\mathbf{T}u)(e^{it}) = \big(P_{\mathbf{H}}(\chi u)\big)(e^{it}) \quad (u\in \mathbf{H} \quad \text{and} \quad \chi(e^{it}) = e^{it}).$$

Here $\mathfrak{E}$ is N-dimensional complex Hilbert space, $H^2(\mathfrak{E})$ is the Hardy space of $\mathfrak{E}$-valued functions on the unit circle, $P_{\mathbf{H}}$ the orthogonal projection of $H^2(\mathfrak{E})$ onto $\mathbf{H}$, and Θ is a matrix-valued "analytic" function, in the sense that $\Theta H^2(\mathfrak{E})\subseteq H^2(\mathfrak{E})$, on the unit circle which is inner from both sides, (i.e., unitary valued a.e. or equivalently, in this case, inner). Finally, the Banach algebras of matrix-valued "analytic" and continuous functions on the unit circle will be denoted by $H^\infty(\mathscr{B}(\mathfrak{E}))$ and $C(\mathscr{B}(\mathfrak{E}))$, respectively. When $\mathfrak{E}$ is simply the complex plane we shall use only H^∞ or C. For further discussion see [10; Chapter IV] and [1; Lectures VII and VIII].

In order to establish our Theorem we need the

Lemma. *If $\psi\in H^\infty$ is a nonconstant inner function which is not a finite Blaschke product then there exists $\varphi\in H^\infty$ such that*

$$\bar{\psi}\varphi\in H^\infty+C \quad \textit{and} \quad \bar{\psi}\varphi^p\notin H^\infty \quad \textit{for any positive integer } p.$$

Proof. This proof is similar to the proofs of Lemma 4 and Lemma 5 in [3]; however, there are some differences so we shall give the details for completeness.

Let $\beta\sigma=\psi$ be the factorization of ψ into a Blaschke product β and a singular inner function σ. If β is nontrivial, then let z_0 be a zero of β of multiplicity m. Define β_0 on the unit circle $\mathscr{T}$ by

$$\beta_0(z) = \left(\frac{z-z_0}{1-\bar{z}_0 z}\right)^m.$$

Then $\varphi=\bar{\beta}_0\psi\in H^\infty$, and $\bar{\psi}\varphi^p=\bar{\beta}_0\varphi^{p-1}$, for any positive integer p. As β_0 does not divide φ^{p-1} we have $\bar{\psi}\varphi^p\notin H^\infty$.

The more difficult case occurs when ψ is purely singular, i.e.

$$\psi(z) = \exp\Big\{-\int_0^{2\pi} h(t,z)\,ds(t)\Big\} \quad (|z|=1), \quad ^{3)}$$

where $h(t,z)=\dfrac{e^{it}+z}{e^{it}-z}$ and s is a singular, finite, positive Borel measure on $[0,2\pi)$. We identify $[0,2\pi)$ with $\mathscr{T}$.

Let $\mathscr{E}$ be a Borel set of Lebesgue measure zero such that $\mathscr{E}$ has full s-measure. By regularity, we can find a closed set $\mathscr{K}$ contained in $\mathscr{E}$ such that $s(\mathscr{K})>0$. Define the measure s_0 on the Borel sets $\mathscr{F}$ in $[0,2\pi)$ by $s_0(\mathscr{F}) = s(\mathscr{K}\cap\mathscr{F})$. Clearly

[3]) Every integral with $h(t,z)$ is interpreted as a limit of the same integral with $h(t;rz)$ as $r\to 1-0$.

s_0 is supported on the closed set $\mathscr{K}$, and the nonconstant inner function

$$\psi_0(z) = \exp\left\{-\int_0^{2\pi} h(t, z)\, ds_0(t)\right\} \quad (|z| = 1)$$

divides ψ. In fact, ψ_0 and $\psi/\psi_0=\gamma$ are relatively prime; therefore, ψ_0 does not divide γ^p for any positive integer p. Since s_0 is supported on $\mathscr{K}$, it follows that ψ_0 is continuous on the complement $\mathscr{T}\setminus\mathscr{K}$. Further, we can choose an outer function v which is continuous on $\mathscr{T}$ and vanishes on $\mathscr{K}$. This follows by applying the portion of the proof on page 80 of [2] in which a log-integrable function $y(\cdot)\geqq 0$ is constructed on $\mathscr{T}$ having the following properties: y is continuous on $\mathscr{T}$, continuously differentiable on $\mathscr{T}\setminus\mathscr{K}$, and vanishing precisely on $\mathscr{K}$. Then we define for $z\in\mathscr{T}$

$$v(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} h(t, z)\log y(e^{it})\, dt\right\};$$

v is an outer function in H^∞ which is continuous on $\mathscr{T}$ and vanishes precisely at the points of $\mathscr{K}$. Set $\varphi=v\gamma$. Again $\varphi\in H^\infty$, and $\bar{\psi}\varphi=\bar{\psi}_0 v$ is continuous. Further, for any positive integer p we have

$$\bar{\psi}\varphi^p = \bar{\psi}_0\gamma^{p-1}v^p,$$

but ψ_0 cannot divide γ^{p-1} because of being relatively prime to ψ, nor can ψ_0 divide v since v is outer; therefore, $\bar{\psi}\varphi^p\notin H^\infty$.

So in each case we have constructed $\varphi\in H^\infty$ such that $\bar{\psi}\varphi\in C$ but $\bar{\psi}\varphi^p\notin H^\infty$ for any positive integer p.

Theorem. *If a weakly closed transitive algebra $\mathscr{A}$ in $\mathscr{B}(\mathfrak{H})$ contains a nonzero $C_0(N)$ operator T, then it is $\mathscr{B}(\mathfrak{H})$.*

Proof. As stated, we shall work within the functional model $\mathbf{T}$; let Θ be the associated inner function. An operator $\mathbf{K}$ on $\mathbf{H}$ commutes with $\mathbf{T}$ if and only if there exists $\Phi\in H^\infty(\mathscr{B}(\mathfrak{E}))$ such that

$$\Phi\Theta H^2(\mathfrak{E}) \subseteqq \Theta H^2(\mathfrak{E})$$

and $\mathbf{K} = \Phi(\mathbf{T})$, where we define

$$\Phi(\mathbf{T})u = P_{\mathbf{H}}(\Phi u)$$

for every $u\in\mathbf{H}$. For the case $N=1$ see [7]; for the general case see [9] and within a functional model [10; in particular Theorem VI.3.6]. Since Θ is unitary valued and $\Phi\Theta H^2(\mathfrak{E})\subseteqq\Theta H^2(\mathfrak{E})$, it follows that $\Phi(\mathbf{T})$ is nonzero if and only if $\Theta^*\Phi\notin H^\infty(\mathscr{B}(\mathfrak{E}))$.

Let $\psi=\det\Theta$ and set $\Psi=\psi\cdot I$, where I is the identity matrix on $\mathfrak{E}$. If ψ is a finite Blaschke product, then $\mathbf{H}$ is finite dimensional and the result follows from Burnside's Theorem [6; Chapter 8]. If ψ is not a finite Blaschke product, then choose, by the lemma, a function $\varphi\in H^\infty$ such that $\psi\varphi\in C$ but $\bar{\psi}\varphi^p\notin H^\infty$ for $p=1, 2, \ldots$. Set

$$\mathbf{H}' = H^2(\mathfrak{E})\ominus\Psi H^2(\mathfrak{E}), \quad \mathbf{T}'u = P_{\mathbf{H}'}(\chi u) \quad \text{and} \quad \Phi(\mathbf{T}')u = P_{\mathbf{H}'}(\Phi u)$$

where $u\in\mathbf{H}'$, $P_{\mathbf{H}'}$ is the orthogonal projection of $H^2(\mathfrak{E})$ onto $\mathbf{H}'$, and $\Phi=\varphi\cdot I$. By the choice of φ we have that

$$\Psi^*\Phi = \bar{\psi}\varphi I\in C(\mathscr{B}(\mathfrak{E})).$$

Further, it is obvious that $\Phi\Psi H^2(\mathfrak{E})\subseteqq\Psi H^2(\mathfrak{E})$ since Φ and Ψ have diagonal matrices as values. Consequently, $\Phi(\mathbf{T}')$ is a compact operator. But $\Phi(\mathbf{T})$ is just the compression of $\Phi(\mathbf{T}')$ to the space $\mathbf{H}$. Hence $\Phi(\mathbf{T})$ is compact too. Further, since $\Phi=\varphi\cdot I$, $\Phi(\mathbf{T})$ is an H^∞ function of T, and hence it is in the weakly closed algebra $\mathscr{A}_{\mathbf{T}}$ generated by $\mathbf{I}$ and $\mathbf{T}$ (see [10; Theorem III.2.1]).

It remains only to show that $\Phi(\mathbf{T})$ is nonzero. This will follow if we can establish that $\Theta^*\Phi\notin H^\infty(\mathscr{B}(\mathfrak{E}))$. Assume the contrary, so that there exists $\Gamma\in H^\infty(\mathscr{B}(\mathfrak{E}))$ such that $\Phi=\Theta\Gamma$. Thus det $\Phi=(\det\Theta)(\det\Gamma)$, so $\bar{\psi}\varphi^N=\det\Gamma\in H^\infty$, a contradiction to the choice of φ. Therefore, $\Phi(\mathbf{T})$ is a nonzero compact operator in $\mathscr{A}_{\mathbf{T}}$. Thus there is a nonzero compact in $\mathscr{A}_T\subseteqq\mathscr{A}$, so by Lomonosov $\mathscr{A}=\mathscr{B}(\mathfrak{H})$.

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References

[1] H. Helson, *Lectures on invariant subspaces*, Academic Press (New York, 1964).
[2] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice Hall (Englewood Cliffs, N. J., 1962).
[3] T. L. Kriete III, B. Moore III, and L. B. Page, Compact intertwining operators, *Michigan Math. J.*, **18** (1971), 115—119.
[4] V. I. Lomonosov, Invariant subspaces of a family of operators commuting with a compact operator, *Funkcional. Anal. Priložen.*, **3** (1973). (Russian)
[5] P. S. Muhly, Compact operators in the commutant of a contraction, *J. Funct. Anal*, **8** (1971), 197—224.
[6] H. Radjávi and P. Rosenthal, *Invariant Subspaces*, Springer-Verlag (Berlin, 1973).
[7] D. Sarason, Generalized interpolation in H^∞, *Trans. Amer. Math. Soc.*, **127** (1967), 179—203.
[8] B. Sz.-Nagy and C. Foiaş, Commutants de certains opérateurs, *Acta Sci. Math.*, **29** (1968), 1—17.
[9] B. Sz. -Nagy and C. Foiaş, Dilatation des commutants d'opérateurs, *C. R. Acad. Sci. Paris*, **266** (1968), 493—495.
[10] B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space*, Akadémiai Kiadó (Budapest, 1970).

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