A note on non-quasitriangular operators*)

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1. Introduction. Let $\mathfrak{H}$ be a fixed, separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathfrak{H})$ denote the algebra of all bounded linear operators on $\mathfrak{H}$. Let $\mathcal{P}$ denote the directed set of all finite rank projections in $\mathcal{L}(\mathfrak{H})$ under the usual ordering, and for each T in $\mathcal{L}(\mathfrak{H})$ define $q(T)=\liminf_{P\in\mathcal{P}}\|(1-P)TP\|$ and $Q(T)=\limsup_{P\in\mathcal{P}}\|(1-P)TP\|$. In [10], HALMOS initiated the study of quasitriangular operators and proved that an operator T is quasitriangular if and only if $q(T)=0$. In [7], DOUGLAS and PEARCY employed the η-function of BROWN and PEARCY (see [5], [12]) to prove that T is a *thin* operator (i.e., an operator that is the sum of a scalar and a compact operator) if and only if $Q(T)=0$. The functions q and Q were studied, respectively, by APOSTOL in [1] and by FOIAŞ and ZSIDÓ in [8]. We appreciatively acknowledge access to preliminary versions of [1] and [8].

In a preliminary version of [8], FOIAŞ and ZSIDÓ proved the following lemma.

Lemma F—Z. *Let T be in $\mathcal{L}(\mathfrak{H})$, $\|T\|=1$, and for $0\leqq t\leqq 1$, let E_t denote the spectral projection of $(T^*T)^{\frac{1}{2}}$ which corresponds to the interval* $[0, t]$. *The following implications are valid.*

i) *If $q(T)=1$, then* $\dim E_t\mathfrak{H}<\aleph_0$ *for all $t<1$.*

ii) *If $q(T)\geqq 0.95$, then there exists $t>1-q(T)$ such that* $\dim E_t\mathfrak{H}<\aleph_0$.

Because of its length and complexity, this writer could not see through the proof of Lemma F—Z. One purpose of this note is to provide (in section 3) a straightforward and short proof of a somewhat stronger version of Lemma F—Z. In particular, we prove that if $\|T\|=1$ and $q(T)>2/3$, then there exists $t>1-q(T)$ such that $\dim E_t\mathfrak{H}<\aleph_0$; an example shows that 2/3 is the best possible lower bound. We discuss the relationship between this result and a theorem of [8]. In section 2, values of q and q/Q are obtained for certain partial isometries. We also prove that if

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A is in $\mathcal{L}(\mathfrak{H})$ and $q(T+A)=0$ for each quasitriangular operator T in $\mathcal{L}(\mathfrak{H})$, then A is a thin operator.

The referee has kindly pointed out that several of the results in section two were proven independently by APOSTOL, FOIAŞ, and ZSIDÓ in [4], and by APOSTOL, FOIAŞ, and VOICULESCU in [2]. These papers followed [1] and [8] in a series of papers on non-quasitriangular operators. In an appendix we give the precise relationship between our results and those of the Rumanian mathematicians.

2. Partial isometries. Let (QT) and (N) denote, respectively, the subsets of quasitriangular and normal operators in $\mathcal{L}(\mathfrak{H})$.

In section 3 of [10], HALMOS proved $(N)\subset(QT)$. For each T in $\mathcal{L}(\mathfrak{H})$ we set $d(T)=\inf_{S\in(QT)}\|T-S\|$ and $d_N(T)=\inf_{S\in(N)}\|T-S\|$. Then clearly $d(T)\leqq d_N(T)$. The proofs of the following two lemmas are easy and will be omitted.

Lemma 2.1. (APOSTOL [1].) *If A and B are operators in $\mathcal{L}(\mathfrak{H})$, then* $|q(A)-q(B)|\leqq\|A-B\|$.

Remark. Lemma 2.1 implies that if T is in $\mathcal{L}(\mathfrak{H})$, then $q(T)\leqq d(T)$. Indeed, if $q(S)=0$, we have $q(T)\leqq\|T-S\|$, and therefore $q(T)\leqq\inf_{S\in(QT)}\|T-S\|$. We are also able to prove the reverse inequality $d(T)\leqq q(T)$ and to thereby conclude that $q(T)$ is the distance from T to the set (QT). This result is not used in this note and the proof will appear elsewhere.

Lemma 2.2. (FOIAŞ and ZSIDÓ [8].) *The following implications are valid.*

i) *If U is a non-unitary isometry, then $q(U)=1$.*

ii) *If T is in $\mathcal{L}(\mathfrak{H})$ and A is a thin operator, then $q(T)=q(T+A)$.*

The following proposition, which we believe to be new, is the converse of Lemma 2.2 ii).

Proposition 2.3. *If A is in $\mathcal{L}(\mathfrak{H})$ and $q(T+A)=0$ for each T in (QT), then A is a thin operator.*

Proof. If A is not thin, then Corollary 3.4 of [5] implies that A is similar to an operator $\mathfrak{H}\oplus\mathfrak{H}$ of the form

$$A_1=\begin{pmatrix} B & V \\ C & 0 \end{pmatrix},$$

where V is a non-unitary isometry. Let A_2 be the operator on $\mathfrak{H}\oplus\mathfrak{H}$ whose matrix is

$$\begin{pmatrix} B & 0 \\ C & 0 \end{pmatrix},$$

and choose an integer $n>1$ such that $n>\|A_2\|$. Let S denote the invertible operator on $\mathfrak{H}\oplus\mathfrak{H}$ of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix},$$

and let $A_3=S^{-1}A_1S$. Finally, let X and Y denote, respectively, the operators on $\mathfrak{H}\oplus\mathfrak{H}$ whose matrices are

$$\begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & nV \\ 0 & 0 \end{pmatrix}.$$

Theorem 6 of [6] implies that $q(X)=0$, and from Lemma 2.2 i), we have $q(X+Y)=n$ Since $|q(X+A_3)-q(X+Y)|\leqq\|A_3-Y\|<n$, it is clear that $q(X+A_3)>0$. Let $R\colon \mathfrak{H}\to\mathfrak{H}\oplus\mathfrak{H}$ be an invertible operator such that $A=R^{-1}A_3R$. Theorem 9 of [6] implies that $q(R^{-1}XR)=0$, and it follows that $q(R^{-1}XR+A)>0$. (Indeed, if $q(R^{-1}XR+A)=0$, another application of [6, Theorem 9] shows that

$$0=q\big(R(R^{-1}XR+A)R^{-1}\big)=q(X+RAR^{-1})=q(X+A_3),$$

which is a contradiction.)

Corollary 2.4. (Douglas and Pearcy [7]) *If A is in $\mathcal{L}(\mathfrak{H})$ and $\lim\limits_{P\in\mathcal{P}}\|(1-P)AP\|=0$, then A is a thin operator.*

Proof. If $\lim\limits_{P\in\mathcal{P}}\|(1-P)AP\|=0$, it is easy to prove that for each T in (QT), $q(A+T)=0$. Then, from Proposition 2.3, A is a thin operator.

Lemma 2.5. *If V is an isometry in $\mathcal{L}(\mathfrak{H})$, then $q(V^*)=0$.*

Proof. The proof is trivial if V is a unilateral shift of multiplicity one. If V is unitary, then V^* is in (N). The proof for an arbitrary isometry procedes from the above special cases via the von Neumann decomposition theorem and Theorem 4 of [10].

Proposition 2.6. *Let V be a partial isometry in $\mathcal{L}(\mathfrak{H})$ with nullity $V=\alpha$ and corank $V=\beta$. The following implications are valid.*

i) *If $\alpha=\beta<\aleph_0$, then $q(V)=0$.*

ii) *If $\alpha=\beta=\aleph_0$, then $q(V)\leqq 1/2$.*

iii) *If $\alpha<\beta$, then $q(V)=1$ and $q(V^*)=0$.*

Proof. i) If $\alpha=\beta<\aleph_0$, there is a finite rank operator F such that $V+F$ is unitary. Then $q(V)=q(V+F)=0$. ii) The proof of [9, Theorem 5] shows that if $\alpha=\beta$, then $d_N(V)\leqq 1/2$. Therefore $q(V)\leqq d(V)\leqq d_N(V)\leqq 1/2$. iii) If $\alpha<\beta$, there is a finite rank operator G such that $V+G$ is a non-unitary isometry. From Lemma 2.2 i), $q(V)=q(V+G)=1$, and from Lemma 2.5, $q(V^*)=q(V^*+G^*)=0$.

Lemma 2.7. *Let U denote a unilateral shift of multiplicity one in $\mathcal{L}(\mathfrak{H})$. If $T=U\oplus 0$ in $\mathcal{L}(\mathfrak{H}\oplus\mathfrak{H})$, then $q(T)=1/2$ and $Q(T)=1$.*

Proof. Let $S=T-1/2$. Since S is bounded below by $1/2$ and nullity $S^*\neq 0$, Lemma 2.1 of [6] implies that $q(T)=q(S)\geqq 1/2$. The reverse inequality follows directly from Proposition 2.6 ii).

Let $\mathcal{P}_1$ denote the directed set of all finite rank projections in $\mathcal{L}(\mathfrak{H}\oplus\mathfrak{H})$ under the usual ordering. To show that $Q(T)\geqq 1$, it suffices to prove that if P_0 is in $\mathcal{P}_1$, then there exists P_1 in $\mathcal{P}_1$ such that $P_1\geqq P_0$ and $\|(1-P_1)TP_1\|=1$. Now since P_0 is in $\mathcal{P}_1$, it is easy to prove that there exist projections R in $\mathcal{P}$ and P_1 in $\mathcal{P}_1$ such that $P_1=R\oplus R$ and $P_1\geqq P_0$. The proof of [6, Lemma 2.1] implies that R may be chosen so that $\|(1-R)UR\|=1$. Then $\|(1-P_1)TP_1\|=\|(1-R)UR\|=1$. Since $Q(T)\leqq\|T\|=1$, the proof is complete.

Proposition 2.8. *If $0\leqq r\leqq 1/2$, there exist partial isometries V and W in $\mathcal{L}(\mathfrak{H}\oplus\mathfrak{H})$ such that $q(V)/Q(V)=r$ and $q(W)=r$.*

Proof. Let U be a unilateral shift of multiplicity one in $\mathcal{L}(\mathfrak{H})$, and for $0\leqq t\leqq 1$ define $P(t)$ by the operator matrix

$$\begin{pmatrix} tU & 0 \\ \sqrt{1-t^2} & 0 \end{pmatrix}.$$

Then $P(t)$ is a norm continuous function on $[0, 1]$ whose values are partial isometries in $\mathcal{L}(\mathfrak{H}\oplus\mathfrak{H})$. It is easy to prove that if $0\leqq t\leqq 1$, then $Q(P(t))>0$. From Lemma 2.1 and an obvious analogue involving Q, the functions q and Q are continuous. If $f_1(t)=q(P(t))$ and $f_2(t)=f_1(t)/Q(P(t))$, then f_1 and f_2 are each continuous on $[0, 1]$ and therefore each has connected range. The proof is completed by noting that $P(0)$ is quasitriangular [6, Theorem 6] and that $f_1(1)=f_2(1)=1/2$ by Lemma 2.7.

3. An improvement of Lemma F—Z. Theorem 3.1. *Let T be in $\mathcal{L}(\mathfrak{H})$, $\|T\|=1$, and for $0\leqq t\leqq 1$, let E_t denote the spectral projection for $(T^*T)^{\frac{1}{2}}$ which corresponds to the interval $[0, t]$. The following implications are valid.*

i) *If $0\leqq t_0<1/3$ and $\dim E_{t_0}=\aleph_0$, then $q(T)\leqq(3-t_0)/4$.*

ii) *If $1/3\leqq t_0<1$ and $\dim E_{t_0}=\aleph_0$, then $q(T)\leqq(1+t_0)/2$.*

Proof. i) Let $T=UP$ denote the polar decomposition of T. Since E_{t_0} reduces P, $P=P_1+P_2$, with P_1 in $\mathcal{L}((E_{t_0}\mathfrak{H})^\perp)$ and P_2 in $\mathcal{L}(E_{t_0}\mathfrak{H})$. Clearly P_1 and P_2 are positive operators. The spectral theorem implies that $\|P_2\|\leqq t_0$ and that $t_0\leqq P_1\leqq 1$. If $V=U(1-E_{t_0})$, then V is a partial isometry such that nullity $V=\aleph_0$. Proposition 2.6 implies that $q(V)\leqq 1/2$, and therefore

$$q(T)\leqq q((1+t_0)/2V)+\|P-(1+t_0)/2(1-E_{t_0})\|\leqq(1+t_0)/4+\|P_1-(1+t_0)/2\oplus P_2\|.$$

Since

$$\|P_1-(1+t_0)/2\| \leqq \sup_{t_0 \leqq t \leqq 1} |t-(1+t_0)/2| = (1-t_0)/2$$

and

$$\|P_2\| \leqq t_0 \leqq (1-t_0)/2,$$

it follows that

$$q(T) \leqq (1+t_0)/4+(1-t_0)/2 = (3-t_0)/4.$$

ii) Proceeding as above, we have $q(T) \leqq q((1-t_0)V)+\|P-(1-t_0)(1-E_{t_0})\| \leqq \leqq (1-t_0)/2+\|(P_1-(1-t_0)) \oplus P_2\|$. Now $\|P_1-(1-t_0)\| \leqq \sup_{t_0 \leqq t \leqq 1} |t-(1-t_0)|$, and an easy calculation shows that the supremum is less than or equal to t_0. Since $\|P_2\| \leqq t_0$, we have $q(T) \leqq (1-t_0)/2+t_0=(1+t_0)/2$.

Corollary 3.2. *Let T be as above. If $q(T)>2/3$, then there exists $t>1-q(T)$ such that* $\dim E_t \mathfrak{H} < \aleph_0$.

Proof. Suppose that for each $t>1-q(T)$, $\dim E_t \mathfrak{H}=\aleph_0$. Since $q(T)>2/3$, then $1-q(T)<1/3$, and therefore $\dim E_{\frac{1}{3}} \mathfrak{H}=\aleph_0$. Theorem 3.1 ii) implies that $q(T) \leqq \leqq (1+1/3)/2=2/3$, which is impossible.

The following example shows that 2/3 is the best possible lower bound for a result like Corollary 3.2.

Example 3.3. Let U denote the unilateral shift of multiplicity one in $\mathscr{L}(\mathfrak{H})$ and let $A=U \oplus -1/3$ and $B=U \oplus 0$. Since $A-1/3$ is bounded below by 2/3 and nullity $(A-1/3)^* \neq 0$, Lemma 2.1 of [6] implies that $q(A)=q(A-1/3) \geqq 2/3$. Lemma 2.7 states that $q(B)=1/2$, and therefore $|q(A)-q(2/3B)|=|q(A)-1/3| \leqq \|A-2/3B\| = =1/3$. Now $1-q(A)=1/3$ and $\dim E_{\frac{1}{3}} \mathfrak{H}=\aleph_0$. Therefore, for each $t>1/3$, $\dim E_t \mathfrak{H} = =\aleph_0$. Since $\|A\|=1$, this example shows that Corollary 3.2 cannot be extended beyond those operators for which $q(T)>2/3\,\|T\|$.

Remark. In [8] Foiaş and Zsidó used Lemma F—Z to prove that if T is in $\mathscr{L}(\mathfrak{H})$, $\|T\|=1$, and $q(T) \geqq 0.95$, then $T=U+S+K$, where U is a nonunitary isometry, S is an operator such that $\|S\|<q(T)$, and K is a finite rank operator. Corollary 3.2 extends this result to any operator T in $\mathscr{L}(\mathfrak{H})$ such that $q(T)>2/3$ and $\|T\|=1$. In particular, T is a semi-Fredholm operator with negative index. We furthur remark that if T is in $\mathscr{L}(\mathfrak{H})$, $\|T\|=1$, and T has the above structure, then $q(T)>1/2$. Indeed, since $T=U+S+K$, $q(T)=q(U+S)$ and therefore $|q(U)-q(T)| \leqq \leqq \|S\|<q(T)$. Since $q(U)=1$, we have $1-q(T)<q(T)$, and the result follows. On the other hand, if $0<\varepsilon \leqq 2/3$, then there exists a Fredholm operator T_ε in $\mathscr{L}(\mathfrak{H} \oplus \mathfrak{H})$, such that $\|T_\varepsilon\|=1$, the index of T_ε is negative, and $q(T_\varepsilon)=\varepsilon$. For example, if V is the unilateral shift of multiplicity one in $\mathscr{L}(\mathfrak{H})$, then we may let T_ε be the operator in $\mathscr{L}(\mathfrak{H}\oplus\mathfrak{H})$ whose matrix is

$$\begin{pmatrix} 0 & V \\ \varepsilon & 0 \end{pmatrix}.$$

Finally, if $1/2<\varepsilon\leqq 2/3$, it is easy to prove that there exists $t>1-q(T_\varepsilon)$ such that $\dim E_t\mathfrak{H}<\aleph_0$. This proves that the converse of Corollary 3.2 is false.

4. Appendix. We wish to indicate that some of our results are related to results in [2] and [4]. (The results in [4] were announced in [3].) Proposition 2.6 is identical to Corollary 2.7 of [4]. The remark on page 3 is contained in Theorem 2.2 of [2], which proves, additionally, that the distance from an operator to the set (QT) is actually attained at some operator in (QT). Lemma 2.7 (about q) is contained in Corollary 4.3 of [2], and Proposition 2.8 (about q) is identical to Theorem 4.4 (about q) of [2]. In each of the above cases the proofs of the corresponding results differ somewhat from one another.

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