The ring N^+ is not adequate

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O. HELMER [2] defined an integral domain R (see [3]) to be *adequate* in case 1) every finitely generated ideal in R is principal and 2) if $a, b \in R$ and $a \neq 0$, then $a = rs$ for $r, s \in R$ such that g.c.d. $(r, b) = 1$ and every nonunit divisor of s shares a nonunit divisor with b. The purpose of this note is to provide a negative answer to the question, raised by J. SZŰCS in the preceding paper [5], of whether or not the ring N^+ of quotients of H^∞ functions by bounded outer functions (see [1]) is adequate. That N^+ satisfies 2) is shown in [5]. That it does not satisfy 1) will be a consequence of the following fact.

Theorem. *There exist finitely generated weak* dense ideals of H^∞ that contain no outer functions.*

Proof. Let a be the atomic inner function

$$a(z) = \exp - [(1+z)/(1-z)],$$

and let b be the Blaschke product with zeros $z_n = 1 - 1/n^2$ $(n = 1, 2, \ldots)$. If I is the ideal in H^∞ generated by a and b, then since a and b have no nontrivial common inner divisors, and since weak* closed ideals of H^∞ have the form φH^∞ for φ inner [4], it follows that I is weak* dense in H^∞.

Suppose I contains an outer function c. Then there exist x and y in H^∞ such that

$$ax + by = c.$$

Letting u be the quotient of the outer factor of x by c, we would then have

$$|a(z_n)u(z_n)| \geqq 1 \tag{1}$$

for $n = 1, 2, \ldots$, which is not possible, as will be shown.

Let P be the Poisson kernel: $P(\theta; z) = \operatorname{Re}[(e^{i\theta}+z)/(e^{i\theta}-z)]$. Since

$$|u(z)| = \exp \frac{1}{2\pi} \int_0^{2\pi} P(\theta; z) \log |u(e^{i\theta})| \, d\theta,$$

taking logarithms converts inequality (1) to

$$(2) \qquad \frac{1}{2\pi}\int_0^{2\pi} P(\theta; z_n)\log|u(e^{i\theta})|\,d\theta-(1+z_n)/(1-z_n) \geqq 0.$$

Choosing $\delta>0$ so that

$$\frac{1}{2\pi}\int_{-\delta}^{\delta}\big|\log|u(e^{i\theta})|\big|\,d\theta<\frac{1}{2}$$

and denoting the left hand side of (2) by d_n, we have

$$d_n \leqq \frac{1}{2\pi}\int_{-\delta}^{\delta} P(\theta; z_n)\big|\log|u(e^{i\theta})|\big|\,d\theta+\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} P(\theta; z_n)\log|u(e^{i\theta})|\,d\theta-(1+z_n)/(1-z_n) \leqq$$

$$\leqq \frac{1+z_n}{1-z_n}\frac{1}{2\pi}\int_{-\delta}^{\delta}\big|\log|u(e^{i\theta})|\big|\,d\theta+\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} P(\theta; z_n)\log|u(e^{i\theta})|\,d\theta-(1+z_n)/(1-z_n) \leqq$$

$$< \frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} P(\theta; z_n)\log|u(e^{i\theta})|\,d\theta-\frac{1}{2}(1+z_n)/(1-z_n).$$

This implies that $d_n \to -\infty$, since the last integral tends to 0 as $n\to\infty$, contradicting (2), and the proof is complete.*)

To see that N^+ does not satisfy 1) consider an ideal I of H^∞ satisfying the conditions of the theorem and generated by functions a and b. The functions a and b then have no common inner divisor and consequently if the ideal they generate in N^+ were principal, then it would have to be all of N^+ since outer functions are units in N^+. Thus we could choose x and y in N^+ such that

$$ax+by=1,$$

and consequently it would follow that the product of the denominators of x and y is in I, which is impossible.

References

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[3] N. Jacobson, *Lectures in abstract algebra*, Vol. I, Van Nostrand (Princeton, N. J., 1951).

[4] T. P. Srinivasan, Simply invariant subspaces and generalized analytic functions, *Proc. Amer. Math. Soc.*, **16** (1965), 813—818.

[5] J. Szűcs, Diagonalization theorems for matrices over certain domains, *Acta Sci. Math.*, **36** (1974), 193—201.

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*) The same example of inner functions a, b for which $ax+by$ is not outer for any choice of $x, y \in H^\infty$, was contained, in connection with another problem, in an earlier letter of C. Foiaş to the Editor. *(The Editor)*