On an ergodic type theorem for von Neumann algebras

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Introduction

The main purpose of this paper is to give a new proof for a theorem of KOVÁCS and SZŰCS [4] concerning the ergodic behaviour of elements in a von Neumann algebra under certain groups of its automorphisms. We shall even point out that a more general result is true for semi-groups of normal endomorphisms instead of groups of automorphisms. In the original proof the Alaoglu—Birkhoff ergodic theorem played a key role, while in our present paper we shall use the Ryll-Nardzewski fixed point theorem [5]. We remark that a different proof also using Ryll-Nardzewski's fixed point theorem is given in [6].

Preliminaries

Let $\mathfrak{H}$ be a complex Hilbert space and denote by $\mathcal{B}(\mathfrak{H})$ the algebra of all bounded linear operators on $\mathfrak{H}$. Among the most often used topologies of $\mathcal{B}(\mathfrak{H})$ are the ultra-strong and ultra-weak topologies. The ultra-strong and the ultra-weak topologies are defined by the semi-norms of the form $T \to \left(\sum_{i=1}^{\infty} \|Tx_i\|^2 \right)^{\frac{1}{2}}$, $T \to \left| \sum_{i=1}^{\infty} (Tx_i, x_i) \right|$, respectively, and where $x_i \in \mathfrak{H}$ and $\sum_{i=1}^{\infty} \|x_i\|^2 < +\infty$.

J. DIXMIER has proved [1] that every ultra-strongly continuous linear form is a linear combination of functionals of the form

$$T \to \sum_{i=1}^{\infty} (Tx_i, x_i), \quad \text{where} \quad x_i \in \mathfrak{H}, \sum_{i=1}^{\infty} \|x_i\|^2 < +\infty. \text{ }^{1})$$

[1]) In particular, we can see that every ultra-strongly continuous linear form is ultra-weakly continuous, as well. Since the ultra-strong operator topology is obviously finer than the ultra-weak one, the converse of this is evident.

A subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathfrak{H})$ that is closed with respect to taking adjoint, contains the identity operator and is closed with respect to one (and then with respect to both)[2]) of the topologies just discussed is called a von Neumann algebra.[3])

By an endomorphism of a von Neumann algebra $\mathscr{A}$ we mean a mapping g of $\mathscr{A}$ into itself that is linear, multiplicative and adjoint preserving. We shall deal with ultra-weakly continuous endomorphisms only. The ultra-weakly continuous endomorphisms can be called normal[4]) endomorphisms, too, relying on a theorem of [2] (Chap. I, § 4. Th. 2, p. 56). In the sequel we shall do so, for the modifier "normal" is shorter than the modifier "ultra-weakly continuous". By the way, we shall never use the above mentioned theorem in our paper, however, the proof of part (iV) of Theorem 2 in [4], which is referred to in our present work, uses a generalization of it.

Let G now be a semi-group of normal endomorphisms of $\mathscr{A}$ and consider an arbitrary but fixed element T of $\mathscr{A}$. Denote by $\mathscr{K}_0(T, G)$ the convex hull of the set of all elements of the form $g(T)$ $(g \in G)$. Let $\mathscr{K}(T, G)$ denote the ultra-strong (and then the ultra-weak) closure of $\mathscr{K}_0(T, G)$. Furthermore, denote by $\mathscr{A}^G$ the set of all elements of $\mathscr{A}$ which are invariant with respect to all elements of G.[5]) Let us denote by $\mathscr{R}(\mathscr{A}, G)$ the set of all ultra-weakly continuous linear forms on $\mathscr{A}$ that are invariant with respect to G. We shall denote by $\mathscr{R}^+(\mathscr{A}, G)$ the positive portion of $\mathscr{R}(\mathscr{A}, G)$.

We shall use in our study the Ryll-Nardzewski fixed point theorem [5]. For the comfort of the reader we state this theorem as a lemma.

Lemma. *Let K be a non-empty weakly compact convex subset of a locally convex Hausdorff space E and let G be a non-contracting*[6]) *semi-group of weakly continuous affine maps of K into itself. Then there exists a common fixed point of the elements of G.*

The following definition of G-finiteness generalizes the one given in [4].

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[2]) From the preceding footnote and from the separation theorem of convex sets ([3], 14.4, p. 119) it follows that every ultra-strongly closed convex subset of $\mathscr{B}(\mathfrak{H})$ is ultra-weakly closed, as well.

[3]) For the theory of von Neumann algebras we refer the reader to [2].

[4]) An endomorphism, or more generally an order preserving positive mapping g of a von Neumann algebra $\mathscr{A}$ into another von Neumann algebra $\mathscr{B}$ is said to be normal if $g(\sup \mathscr{F}) = \sup g(\mathscr{F})$ for any upward dierected bounded subset $\mathscr{F}$ of the positive portion of $\mathscr{A}$.

[5]) In general, $\mathscr{A}^G$ is not a von Neumann algebra but there exists a maximal (orthogonal) projection P in $\mathscr{A}^G$ such that $\mathscr{A}^G | P\mathfrak{H}$ is a von Neumann algebra.

[6]) By definition, G is non-contracting if for any two distinct elements x and y of K there exists a strongly continuous semi-norm p on E (depending on x and y) such that $\inf \{p(gx-gy) : g \in G\} > 0$.

Definition. Let $\mathscr{A}$ be a von Neumann algebra and consider a semi-group G of normal endomorphisms of $\mathscr{A}$. The algebra $\mathscr{A}$ is said to be *G-finite* if for every non-zero element T of $\mathscr{A}^{+}$[7]) there exists an element σ of $\mathscr{R}^{+}(\mathscr{A}, G)$ such that $\sigma(T)\neq 0$.[8])

The theorems

Kovács and Szűcs [4] proved the following:[9])

Theorem 1. *Let $\mathscr{A}$ be a von Neumann algebra and consider a semi-group G of its normal endomorphisms. Suppose that $\mathscr{A}$ is G-finite. Then for every element T of $\mathscr{A}$ the set $\mathscr{K}(T, G)\cap\mathscr{A}^G$ consists of exactly one element.*

Proof. The von Neumann algebra $\mathscr{A}$ with the ultra-strong operator topology is a locally convex Hausdorff space. By Dixmier's result cited in Preliminaries, the weak topology of this locally convex space coincides with the ultra-weak operator topology. It is a well-known and easily provable fact that the unit ball of $\mathscr{B}(\mathfrak{H})$ is ultra-weakly compact. This implies that $\mathscr{K}(T, G)$ is compact in the ultra-weak operator topology for every element T of $\mathscr{A}$. For every $g\in G$ we obviously have $g(\mathscr{K}_0(T, G))\subseteqq\mathscr{K}_0(T, G)$ and then by the ultra-weak continuity of the elements of G we have $g(\mathscr{K}(T, G))\subseteqq\mathscr{K}(T, G)$. Ryll-Nardzewski's theorem shows that to prove $\mathscr{K}(T, G)\cap\mathscr{A}^G\neq\emptyset$ for any $T\in\mathscr{A}$ it is enough to show that G is non-contracting on every $\mathscr{K}(T, G)$ in the ultra-strong operator topology. To verify this, fix an element T of $\mathscr{A}$ and consider two distinct members A and B of $\mathscr{K}(T, G)$. From the G-finiteness of $\mathscr{A}$ there follows the existence of an element σ of $\mathscr{R}^{+}(\mathscr{A}, G)$ such that $\sigma((A-B)^*(A-B))\neq 0$. For every element S of $\mathscr{A}$ put $p(S)=[\sigma(S^*S)]^{\frac{1}{2}}$. It is easy to see that p is a semi-norm on $\mathscr{A}$. Furthermore, for every element g of G, we have

$$p^2(g(A)-g(B)) = p^2(g(A-B)) = \sigma(g(A-B)^*(g(A-B))) =$$
$$= \sigma(g(A-B)^*(A-B)) = \sigma((A-B)^*(A-B)).$$

This shows that $\inf\{p(g(A)-g(B)) : g\in G\}>0$. We shall show that p is ultra-strongly continuous. In fact, consider a net $\{S_\alpha\}$ of elements of $\mathscr{A}$ that tends to 0 in the ultra-strong topology. Then, by the definitions of the ultra-strong and ultra-weak topologies, $S_\alpha^* S_\alpha$ tends to 0 in the ultra-weak topology. Hence $p(S_\alpha)=[\sigma(S_\alpha^* S_\alpha)]^{\frac{1}{2}}$ tends to 0 wich shows that p is ultra-strongly continuous. Summarizing all our investigations,

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7) $\mathscr{A}^{+}$ denotes the positive portion of $\mathscr{A}$.

8) If $\mathscr{A}$ as G-finite, then it is easy to see that $g(I)=I$ for every element g of G. Therefore in this case $\mathscr{A}^G$ is a von Neumann algebra (see the footnote on p. 000).

9) They supposed that G was a group.

the Ryll-Nardzewski fixed point theorem applies to every $\mathscr{K}(T, G)$ and to the semi-group G, so $\mathscr{K}(T, G)\cap\mathscr{A}^G\neq\emptyset$ for every element T of $\mathscr{A}$.

To accomplish the proof of Theorem 1 we have to show that $\mathscr{K}(T, G)\cap\mathscr{A}^G$ has only one element. To this effect denote by Q the set of all linear maps of $\mathscr{A}$ into itself of the form $\sum_{i=1}^{n}\alpha_i g_i \left(\alpha_i>0,\ \sum_{i=1}^{n}\alpha_i=1,\ g_i\in G\right)$. Consider a fixed element T of $\mathscr{A}$ and suppose that S and R are two distinct elements of $\mathscr{K}(T, G)\cap\mathscr{A}^G$. Since $S\in\mathscr{K}(T, G)$, there exists a net $\{g_\alpha\}$ of elements of Q such that $\lim_\alpha g_\alpha(T)=S$ where the limit is taken in the ultra-weak topology. For every element σ of $\mathscr{R}^+(\mathscr{A}, G)$ we have

$$\sigma\big((S-R)^*S\big) = \lim_\alpha \sigma\big((S-R)^*g_\alpha(T)\big) = \lim_\alpha \sigma\big(g_\alpha((S-R)^*T)\big) = \sigma\big((S-R)^*T\big).$$

Similarly, for R in place of S we have

$$\sigma\big((S-R)^*R\big) = \sigma\big((S-R)^*T\big).$$

By subtraction we obtain

$$\sigma\big((S-R)^*(S-R)\big) = 0.$$

Since σ was an arbitrary element of $\mathscr{R}^+(\mathscr{A}, G)$, the G-finiteness of $\mathscr{A}$ implies that $S=R$. This completes the proof of Theorem 1.

In accordance with [4] let us denote the unique element of $K(T, G)\cap\mathscr{A}^G$ by T^G.

Relying on the previous theorem, Kovács and Szűcs [4] proved the following result stated for groups of automorphisms only.

Theorem 2. *Let $\mathscr{A}$ be a von Neumann algebra in a complex Hilbert space $\mathfrak{H}$ and let G be a semi-group of normal endomorphisms of $\mathscr{A}$. Suppose that $\mathscr{A}$ is G-finite. Then the mapping $T\to T^G$ possesses the following properties:*

(i) $\sigma(T)=\sigma(T^G)$ *for every* $\sigma\in\mathscr{R}(\mathscr{A}, G)$ *and* $T\in\mathscr{A}$;

(ii) $T\to T^G$ *is linear and strictly positive;*

(iii) $(ST)^G=ST^G$ *and* $(TS)^G=T^G S$ *for* $T\in\mathscr{A}$, $S\in\mathscr{A}^G$;

(iv) $T\to T^G$ *is ultra-weakly and ultra-strongly continuous;*

(v) $T=T^G$ *for every* $T\in\mathscr{A}^G$;

(vi) $\big(g(T)\big)^G=T^G$ *for every* $T\in\mathscr{A}$ *and* $g\in G$.

Conversely, if we do not suppose that $\mathscr{A}$ is G-finite but we know that there exists an ultra-weakly continuous strictly[10]) *positive linear mapping $T\to T'$ of $\mathscr{A}$ onto $\mathscr{A}^G$ such that*

[10]) In [4] the assumption of strictness does not occur.

a) *$T=T'$ for every $T\in\mathcal{A}^G$;*

b) *$(g(T))'=T'$ for every $T\in\mathcal{A}$, $g\in G$,*

then $\mathcal{A}$ is necessarily G-finite and $T'=T^G$ for every $T\in\mathcal{A}$.

Relying on our Theorem 1, in the more general situation of semi-groups of normal endomorphisms properties (i)—(vi) of the so-called G-canonical mapping $T\to T^G$ can be proved in the same way as they were in [4] with a minor modification in the proof of (vi) except for the first statement in property (ii) which asserts that the mapping $T\to T^G$ is linear. The proof of this fact in [4] relies not only on Theorem 1 of [4] but on its proof as well. Now we are going to show the linearity of the G-canonical mapping in the more general situation when G is a semi-group of normal endomorphisms of $\mathcal{A}$.

In fact, suppose that $\mathcal{A}$ is G-finite and use the notations of Theorem 1. Consider two elements, R and S, of $\mathcal{A}$. Since the G-canonical map is obviously homogeneous, it is enough to show that $(R+S)^G=R^G+S^G$. Since $(R+S)^G\in\mathcal{K}(R+S, G)$ we can find a net $\{g_\alpha\}$ of elements of Q such that

$$(R+S)^G = \text{ultra-weak} \lim_\alpha g_\alpha(R+S). \tag{1}$$

Since $\mathcal{K}(R, G)$ is ultra-weakly compact, we can find a subnet $\{h_\beta\}$ of the net $\{g_\alpha\}$ such that $h_\beta(R)$ is convergent in the ultra-weak topology. Then (1) shows that $h_\beta(S)$ is ultra-weakly convergent, too. Put $R_0=\lim_\beta h_\beta(R)$ and $S_0=\lim_\beta h_\beta(S)$. Then we have

$$R_0\in\mathcal{K}(R, G), \quad S_0\in\mathcal{K}(S, G) \quad \text{and} \quad (R+S)^G = R_0+S_0.$$

The fact $R_0\in\mathcal{K}(R, G)$ implies that $\mathcal{K}(R_0, G)\subseteq\mathcal{K}(R, G)$ and so, by uniqueness, $R_0^G=R^G$. Similarly, $S_0^G=S^G$. Choose a net $\{k_\gamma\}$ of elements of Q such that ultra-weak $\lim_\gamma k_\gamma(R_0)=R^G$. Then we have

$$k_\gamma(S_0) = k_\gamma((R+S)^G-R_0) = (R+S)^G-k_\gamma(R_0)$$

which shows that $k_\gamma(S_0)$ is convergent in the ultra-weak topology, too. Put $\lim_\gamma k_\gamma(S_0)=S_1$. Then we have

$$(R+S)^G = R^G+S_1, \quad S_1\in\mathcal{K}(S, G).$$

The fact $S_1\in\mathcal{K}(S, G)$ implies that $S_1^G=S^G$. Choose a net $\{l_\delta\}$ of elements of Q such that ultra-weak $\lim_\delta l_\delta(S_1)=S^G$. Then we have

$$(R+S)^G = \lim_\delta l_\delta((R+S)^G) = \lim_\delta l_\delta(R^G)+\lim_\delta l_\delta(S_1) = R^G+S^G.$$

This completes the proof of the linearity of the G-canonical mapping.

As far as the rest of Theorem 2 is concerned, in the case of semi-groups of normal endomorphisms we have to modify the proof of [4] in the following way:

Suppose that $\mathscr{A}$ admits an ultra-weakly continuous strictly positive linear mapping $T \to T'$ having properties a) and b) of Theorem 2. Consider an arbitrary non-vanishing element S of $\mathscr{A}^+$. It follows that S' is a non-vanishing positive element of $\mathscr{A}^G$. Then put $T_0 = S'$ and define σ as in [4]. We have $\sigma \in \mathscr{R}^+(\mathscr{A}, G)$ and $\sigma(S) = \sigma(S') = \sigma(T_0) \neq 0$. Since S was an arbitrary non-vanishing element of $\mathscr{A}^+$, this shows that $\mathscr{A}$ is G-finite.

The equation $T' = T^G$ $(T \in \mathscr{A})$ can be proved in the same way as in [4].

We are now going to conclude with an example of a von Neumann algebra $\mathscr{A}$ and a cyclic semi-group $G = \{g^n\}_{n=1}^{\infty}$ of its normal endomorphisms such that $\mathscr{A}$ is G-finite and g is not an automorphism. In fact, let $\mathscr{A}$ be the von Neumann algebra of all multiplication operators generated by essentially bounded Lebesgue measurable functions on the complex Hilbert space $L^2[0, 1]$ and let g be the endomorphism of $\mathscr{A}$ generated by the point transformation $T: x \to 2x \pmod 1$ in the following way: $[g(f)](x) = f(Tx)$ $(f \in \mathscr{A})$ (here we identified the elements of $\mathscr{A}$ with any of the functions which generate them). It is immediate that g is normal, $\mathscr{A}$ is G-finite and g is not an automorphism. In this case the G-canonical mapping of $\mathscr{A}$ reduces to the mapping $f \to \left(\int_0^1 f(t)\,dt\right) e$ where e denotes the constant 1 function on $[0, 1]$.

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