A spectral characterization of the maximal ideal in factors

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Introduction. In a recent paper ([3]) J. A. DYER, P. PORCELLI and M. ROSENFELD obtained a spectral characterization of the elements in the greatest proper ideal $\mathscr{J}$ of a properly infinite factor $\mathscr{M}$, namely that $x\in\mathscr{J}$ iff $\sigma(x+b)\cap\sigma(b)\neq\emptyset$ for every $b\in\mathscr{M}$. On the other hand, they proved that if $\mathscr{M}$ is a factor of type I_n, $n<\infty$, then for any $0\neq x\in\mathscr{M}$ there is $b\in\mathscr{M}$ such that $\sigma(x+b)\cap\sigma(b)=\emptyset$ and they conjectured that the same assertion is true if $\mathscr{M}$ is a factor of type II_1.

In the present paper we prove this conjecture by showing that if $\mathscr{M}$ is a factor of type II_1 and $0\neq x\in\mathscr{M}$, then there is a nilpotent element $b\in\mathscr{M}$ such that $x+b$ is invertible (Corollary 4), getting exactly the same result as for factors of type I_n, $n<\infty$. Moreover, the same result is established for elements in a properly infinite factor $\mathscr{M}$, which are not of the form $\lambda+a$ with $\lambda\in\mathbf{C}$ and $a\in\mathscr{J}$ (that is, for not "thin" elements; Corollary 5). This is done by proving Theorem 2 below, which allows us to represent every element x in $\mathscr{M}$ as a suitable operator matrix and then by using the trick of BROWN and HALMOS (cf. the proof of Theorem C in [3] and also below, Remark 3).

For results concerning operator algebras we refer to the treatise DIXMIER [2].

Two projections e, f in a C^*-algebra $\mathscr{M}$ are said to be *equivalent*, $e\sim f$, if there is an element $v\in\mathscr{M}$ such that $v^*v=e$, $vv^*=f$; then $v=ve=fv$ and e, f belong to the same (two sided) ideals of $\mathscr{M}$. If $\mathscr{M}$ is a W^*-algebra and $x\in\mathscr{M}$, then $LP(x)$ (resp. $RP(x)$) means the left projection (resp. the right projection) of x; it is known that $LP(x)\sim RP(x)$ ("$LP\sim RP$" theorem) (cf. [6]). By $\mathscr{B}(\mathfrak{H})$ we denote the algebra of all (bounded) operators on the Hilbert space $\mathfrak{H}$. As usually, for an element x in a Banach algebra we denote by $\sigma(x)$ its spectrum.

1. We begin with the following lemma which is surely known:

Lemma. *Let $e, f\in\mathscr{B}(\mathfrak{H})$ be two projections such that there is $0<\lambda\leqq 1$ with $f-fef\geqq\lambda f$. Then $e\wedge f=0$ and $(e\vee f)(\mathfrak{H})=e(\mathfrak{H})+f(\mathfrak{H})$.*

Proof. The inequality $f-fef\geqq\lambda f$ is equivalent to the conditions $\|(1-e)f\xi\|\geqq$ $\geqq\sqrt{\lambda}\,\|f\xi\|$, $\xi\in\mathfrak{H}$. If $e\xi=\xi=f\xi$ then it follows that $\xi=0$, whence $e(\mathfrak{H})\cap f(\mathfrak{H})=0$

and $e\wedge f=0$. For $\xi,\eta\in\mathfrak{H}$ we have $\|e\eta+f\xi\|=\|(e\eta+ef\xi)+(1-e)f\xi\|\geqq\|(1-e)f\xi\|\geqq$ $\geqq\sqrt{\lambda}\,\|f\xi\|$, so that, if $\zeta\in(e\vee f)(\mathfrak{H})=\overline{e(\mathfrak{H})+f(\mathfrak{H})}$ and $\zeta=\lim_n(e\eta_n+f\xi_n)$ then the sequence $\{f\xi_n\}$ is convergent and $\xi=\lim_n f\xi_n\in f(\mathfrak{H})$. Then $\zeta-\xi=\lim_n e\eta_n\in e(\mathfrak{H})$, i.e. $\zeta\in e(\mathfrak{H})+f(\mathfrak{H})$. Hence $(e\vee f)(\mathfrak{H})=e(\mathfrak{H})+f(\mathfrak{H})$. Q.E.D.

Remark. The greatest λ satisfying the inequality $f-fef\geqq\lambda^2 f$ could be called "the sinus of the angle between the projections e and f". If $\lambda=1$, then e and f are orthogonal, and the lemma says that if "the angle between e and f" is not zero, then $(e\vee f)(\mathfrak{H})=e(\mathfrak{H})+f(\mathfrak{H})$.

2. Theorem. *Let $\pi\colon\tilde{\mathcal{M}}\to\mathcal{M}$ be a representation of the W^*-algebra $\tilde{\mathcal{M}}$ on the C^*-algebra $\mathcal{M}$, $\mathcal{Z}$ the center of $\mathcal{M}$ and $\mathcal{J}\subset\mathcal{M}$ a closed ideal in $\mathcal{M}$. For every $x\in\mathcal{M}$, $x\notin\mathcal{Z}+\mathcal{J}$ there are: an invertible element $u\in\mathcal{M}$, two equivalent orthogonal projections $e_1,e_2\in\mathcal{M}$, $e_1\notin\mathcal{J}\not\ni e_2$, and an element $y\in e_1\mathcal{M}e_2$ such that, putting $x_0=u^{-1}xu$, we have:*

(i) $x_0e_1=e_2x_0e_1,\ yx_0e_1=e_1,\ x_0y=e_2$;

(ii) *for every projection $e_1'\leqq e_1$ there is an equivalent projection $e_2'\leqq e_2$ and an element $y'\in e_1'\mathcal{M}e_2'$ such that*

$$x_0e_1'=e_2'x_0e_1',\quad y'x_0e_1'=e_1',\quad x_0y'=e_2'.$$

Proof. Suppose the W^*-algebra $\tilde{\mathcal{M}}$ is realized as a von Neumann algebra acting on the Hilbert space $\mathfrak{H}$, $\tilde{\mathcal{M}}\subset\mathcal{B}(\mathfrak{H})$ and put $\tilde{\mathcal{J}}=\pi^{-1}(\mathcal{J})$. Since the center of $\mathcal{M}/\mathcal{J}$ is the canonical image of $\mathcal{Z}$ and $x\notin\mathcal{Z}+\mathcal{J}$, there is a projection $p\in\mathcal{M}$ such that $(1-p)xp\notin\mathcal{J}$. Let $\tilde{x}\in\tilde{\mathcal{M}}$ and let $\tilde{p}\in\tilde{\mathcal{M}}$ be a projection such that $\pi(\tilde{x})=x$, $\pi(\tilde{p})=p$ and put $\tilde{a}=(\tilde{1}-\tilde{p})\tilde{x}\tilde{p}$, $|\tilde{a}|=\sqrt{\tilde{a}^*\tilde{a}}$. Since the support of $|\tilde{a}|$ is smaller than $\tilde{p}$ and $|\tilde{a}|\notin\tilde{\mathcal{J}}$, by using the spectral theorem we get a spectral projection $\tilde{e}\notin\tilde{\mathcal{J}}$, $\tilde{e}\leqq\tilde{p}$ such that

$$\|\tilde{a}\tilde{e}\xi\|\geqq\lambda\|\tilde{e}\xi\|;\quad \xi\in\mathcal{H};\quad \lambda>0.$$

In particular, for $\xi\in\mathfrak{H}$:

$$\|\tilde{x}\tilde{e}\xi\|\geqq\|(1-\tilde{e})\tilde{x}\tilde{e}\xi\|\geqq\|(1-\tilde{p})\tilde{x}\tilde{e}\xi\|=\|\tilde{a}\tilde{e}\xi\|\geqq\lambda\|\tilde{e}\xi\|.$$

We have $\tilde{e}=RP(\tilde{x}\tilde{e})$ and we put $\tilde{f}=LP(\tilde{x}\tilde{e})$. Then $\tilde{f}(\mathcal{H})=\overline{\tilde{x}\tilde{e}(\mathfrak{H})}=\tilde{x}\tilde{e}(\mathfrak{H})$ and $\|(1-\tilde{e})\tilde{x}\tilde{e}\xi\|\geqq(\lambda/\|\tilde{x}\|)\|\tilde{x}\tilde{e}\xi\|$, $\xi\in\mathfrak{H}$; that is $\tilde{f}-\tilde{f}\tilde{e}\tilde{f}\geqq(\lambda/\|\tilde{x}\|)^2\tilde{f}$. Hence $\tilde{e}\wedge\tilde{f}=0$ and $(\tilde{e}\vee\tilde{f})(\mathfrak{H})=\tilde{e}(\mathcal{H})+\tilde{f}(\mathcal{H})$, by Lemma 1.

The operator

$$(\tilde{1}-\tilde{e})\tilde{f}\colon\ \tilde{f}(\mathfrak{H})\to(\tilde{1}-\tilde{e})\tilde{f}(\mathfrak{H})$$

is invertible. We put $\tilde{e}_1=\tilde{e}$, $\tilde{e}_2=LP\big((\tilde{1}-\tilde{e})f\big)$ and we note that $\tilde{f}=RP\big((\tilde{1}-\tilde{e})\tilde{f}\big)$. Let $(\tilde{1}-\tilde{e})\tilde{f}=\tilde{w}\,|(\tilde{1}-\tilde{e})\tilde{f}|$ be the polar decomposition of $(\tilde{1}-\tilde{e})\tilde{f}$ and let $\tilde{g}\in\tilde{f}\tilde{\mathcal{M}}\tilde{f}$ be the inverse of $|(\tilde{1}-\tilde{e})\tilde{f}|$ in $\tilde{f}\tilde{\mathcal{M}}\tilde{f}$. Then the operator

$$\tilde{g}\tilde{w}^*\colon\ \tilde{e}_2(\mathfrak{H})\to\tilde{f}(\mathfrak{H})$$

is the inverse of the operator:

$$(\tilde{1}-\tilde{e})\tilde{f}:\ \tilde{f}(\mathfrak{H})\to\tilde{e}_2(\mathfrak{H}).$$

Now define $\tilde{u}=\tilde{g}\tilde{w}^*\tilde{e}_2+(\tilde{1}-\tilde{e}_2)\in\tilde{\mathcal{M}}$. The operator $\tilde{u}$ is one-to-one and $\tilde{u}(\mathfrak{H})=$ $=\tilde{f}(\mathfrak{H})+\tilde{e}(\mathfrak{H})+(\tilde{1}-\tilde{e}_1-\tilde{e}_2)(\mathfrak{H})$. Since $(\tilde{e}\vee\tilde{f})(\mathfrak{H})=\tilde{e}(\mathfrak{H})+\tilde{f}(\mathfrak{H})$ it follows that $\tilde{u}(\mathfrak{H})=\mathfrak{H}$, whence $\tilde{u}$ is invertible and $\tilde{u}^{-1}\in\tilde{\mathcal{M}}$ by the closed graph theorem.

We put $\tilde{x}_0=\tilde{u}^{-1}\tilde{x}\tilde{u}$. The operator

$$\tilde{x}\tilde{e}:\ \tilde{e}(\mathfrak{H})\to\tilde{f}(\mathfrak{H})$$

is invertible, thus so is the operator

$$\tilde{x}_0\tilde{e}_1:\ \tilde{e}_1(\mathfrak{H})\to\tilde{e}_2(\mathfrak{H})$$

as well as any operator:

$$\tilde{x}_0\tilde{e}_1':\ \tilde{e}_1'(\mathfrak{H})\to\tilde{e}_2'(\mathfrak{H})$$

where $\tilde{e}_1'\leqq\tilde{e}_1$ is a projection and $\tilde{e}_2'=LP(\tilde{x}_0\tilde{e}_1')\leqq\tilde{e}_2$. It follows that $\tilde{x}_0\tilde{e}_1'=\tilde{e}_2'\tilde{x}_0\tilde{e}_1'$ and that there is an element $\tilde{y}'\in\tilde{e}_1'\tilde{M}\tilde{e}_2'$, such that $\tilde{y}'\tilde{x}_0\tilde{e}_1'=\tilde{e}_1'$, $\tilde{x}_0\tilde{y}'=\tilde{e}_2'$. By the "$LP\sim RP$" theorem we have $\tilde{e}_1=\tilde{e}\sim\tilde{f}\sim\tilde{e}_2$ and $\tilde{e}_1'\sim\tilde{e}_2'$. Furthermore, since $\tilde{e}\notin\tilde{\mathcal{J}}$, we also have $\tilde{e}_1\notin\tilde{\mathcal{J}}\not\ni\tilde{e}_2$.

Putting $u=\pi(\tilde{u})$, $e_1=\pi(\tilde{e}_1)$, $e_2=\pi(\tilde{e}_2)$ and $y=\pi(\tilde{y})$ ($\tilde{y}$ is the corresponding $\tilde{y}'$ for $\tilde{e}_1'=\tilde{e}_1$) we obtain (i). If $e_1'\leqq e_1$ is a projection then there is a projection $\tilde{e}_1'\leqq\tilde{e}_1$ such that $\pi(\tilde{e}_1')=e_1'$ and (ii) follows. Q.E.D.

3. Remark. Brown and Halmos proved that for every $0\neq x\in\mathcal{B}(\mathfrak{H})$, dim $\mathfrak{H}<\infty$, there is a nilpotent element $b\in\mathcal{B}(\mathfrak{H})$ such that $x+b$ is invertible (cf. the proof of Theorem C in [3]). The first step of their proof consists of finding an element in $\mathcal{B}(\mathfrak{H})$ similar to x and with a suitable matrix form; this suggested us Theorem 2. The second step of their proof is as follows. Let x be an operator $(n\times n)$-matrix of the "suitable form" $x=(x_{ij})$ with $x_{j,1}=0$ for $j<n$ and $x_{n,1}$ invertible. Consider the matrix $b=(b_{ij})$ with $b_{i,i+1}=1-x_{i,i+1}$, $b_{i,j}=-x_{i,j}$ for $j>i+1$ and $b_{i,j}=0$ for $i\geqq j$. For $n=3$ the picture is as follows

$$x=\begin{pmatrix}0 & x_{12} & x_{13}\\ 0 & x_{22} & x_{23}\\ x_{31} & x_{32} & x_{33}\end{pmatrix},\quad b=\begin{pmatrix}0 & 1-x_{12} & -x_{13}\\ 0 & 0 & 1-x_{23}\\ 0 & 0 & 0\end{pmatrix},\quad x+b=\begin{pmatrix}0 & 1 & 0\\ 0 & x_{22} & 1\\ x_{31} & x_{32} & x_{33}\end{pmatrix}.$$

It is obvious that b is nilpotent and $x+b$ is invertible.

Now, consider a C^*-algebra $\mathcal{M}$ with unit such that there are n mutually equivalent, mutually orthogonal projections $e_1,\dots,e_n$ in $\mathcal{M}$ such that $e_1+\dots+e_n=1$. Then every element $x\in\mathcal{M}$ can be represented as an operator $(n\times n)$-matrix whose components are in $e_1\mathcal{M}e_1$. Namely, let v_i be an element of $\mathcal{M}$, such that $v_i^*v_i=e_1$,

$v_i v_i^* = e_i$, $v_i = v_i e_1 = e_i v_i$, for $i = 1, \ldots, n$. Put $x_{ij} = v_i^* x v_j \in e_1 \mathcal{M} e_1$. Then:

$$x = \sum_{i,j} e_i x e_j = \sum_{i,j} v_i x_{ij} v_j^* = (x_{ij})$$

where the last equality is a notation. We say that x_{ij} is the (i, j)-th component in the matrix representation of x with respect to the "basis" $(e_1, \ldots, e_n)$. It is easy to see that $(x^*)_{i,j} = x_{j,i}^*$ and $(xy)_{i,j} = \sum_k x_{ik} y_{kj}$. In particular, if x has a "suitable" matrix representation then there is a nilpotent element b in $\mathcal{M}$ such that $x + b$ is invertible.

The method just explained and Theorem 2 allows us to settle affirmatively the conjecture of Dyer, Porcelli and Rosenfeld.

4. Corollary. *Let $\mathcal{M}$ be a finite factor and $0 \neq x \in \mathcal{M}$. There is a nilpotent element $b \in \mathcal{M}$ such that $x + b$ is invertible. In particular $\sigma(x+b) \cap \sigma(b) = \emptyset$.*

Proof. For factors of type I_n, $n < \infty$ the result is known. So, let $\mathcal{M}$ be a factor of type II_1 and denote by d its relative dimension function $(d(1) = 1)$. If x is a scalar element, then we may take $b = 0$. If not, x is not a central element. Since it suffices to prove the assertion of the corollary for an element similar (in $\mathcal{M}$) to x, from Theorem 2 it follows that we can suppose that there are: a positive integer n, two equivalent orthogonal projections $\bar{e}_1, \bar{e}_2 \in \mathcal{M}$ and an element $y \in \bar{e}_1 \mathcal{M} \bar{e}_2$ such that: $d(\bar{e}_1) = d(\bar{e}_2) = 1/n$, $x\bar{e}_1 = \bar{e}_2 x \bar{e}_1$, $yx\bar{e}_1 = \bar{e}_1$, $xy = \bar{e}_2$. Let $e_1, \ldots, e_n \in \mathcal{M}$ be mutually equivalent, mutually orthogonal projections with $e_1 + \ldots + e_n = 1$ and $e_1 = \bar{e}_1$, $e_n = \bar{e}_2$; then $xe_1 = e_n x e_1$, $y \in e_1 \mathcal{M} e_n$, $yxe_1 = e_1$, $xy = e_n$. The matrix representation of x with respect to the "basis" $(e_1, \ldots, e_n)$ is a "suitable" one. Indeed, for $j \neq n$:

$$x_{j,1} = v_j^* x v_1 = v_j^* e_j x e_1 v_1 = v_j^* e_j e_n x e_1 v_1 = 0$$

and $x_{n,1}$ is invertible in $e_1 \mathcal{M} e_1$ with the inverse $y_{1,n}$:

$$y_{1,n} x_{n,1} = (v_1^* y v_n)(v_n^* x v_1) = v_1^* y e_n x v_1 = v_1^* y x v_1 = v_1^* e_1 v_1 = e_1$$

Hence the corollary follows from Remark 3. Q.E.D.

We can also extend the result obtained in [3] for properly infinite factors to properly infinite "*C^*-factors*". In the following corollary "*large*" projections are those which are equivalent to 1 and "*small*" projections are those which are not equivalent to 1.

5. Corollary. *Let $\pi : \tilde{\mathcal{M}} \to \mathcal{M}$ be a representation of the properly infinite W^*-algebra $\tilde{\mathcal{M}}$ on the C^*-algebra $\mathcal{M}$ whose center reduces to the scalar elements. Then $\mathcal{M}$ has a greatest ideal $\mathcal{J}$ which is generated by the small projections in $\mathcal{M}$, and an element $x \in \mathcal{M}$ belongs to $\mathcal{J}$ iff $\sigma(x+b) \cap \sigma(b) \neq \emptyset$ for every $b \in \mathcal{M}$. Moreover, if x is not* "thin" *(i.e. x is not of the form $\lambda + a$ where $\lambda \in \mathbf{C}$ and $a \in \mathcal{J}$), then there is a nilpotent element $b \in \mathcal{M}$ such that $x + b$ is invertible.*

Proof. It is well known that in a properly infinite factor the small projections form a p-ideal and since any factor has a greatest ideal this is generated by the small projections. It is easy to imitate the above argument in the present situation and so, $\mathcal{M}$ has a greatest ideal $\mathcal{J}$ and $\mathcal{J}$ is generated by the small projections.

Since the spectral theorem holds in $\mathcal{M}$ and in $\mathcal{M}$ there are "large" and "small" projections, HALMOS's proof for the infinite dimensional case of the theorem of DYER, PORCELLI and ROSENFELD applies and so, $x\in\mathcal{J}$ iff $\sigma(x+b)\cap\sigma(b)\neq\emptyset$ for every $b\in\mathcal{M}$.

Now suppose $x\in\mathcal{M}$ and x is not "thin". By Theorem 2 we can suppose that there are two orthogonal equivalent projections $\bar{e}_1$, $\bar{e}_2\in\mathcal{M}$, $\bar{e}_1\notin\mathcal{J}\not\ni\bar{e}_2$ and an element $y\in\bar{e}_1\mathcal{M}\bar{e}_2$ such that $x\bar{e}_1=\bar{e}_2x\bar{e}_1$, $yx\bar{e}_1=\bar{e}_1$, $xy=\bar{e}_2$. Since $1-\bar{e}_1\geqq\bar{e}_2\sim 1$ and $\bar{e}_1\sim 1= =\bar{e}_1+(1-\bar{e}_1)$ there are two orthogonal large projection $\bar{e}_1'$, $\bar{e}''$ whose sum is $\bar{e}_1$. Again by Theorem 2, we find a projection $\bar{e}_2'\leqq\bar{e}_2$ and an element $y'\in\bar{e}_1'\mathcal{M}\bar{e}_2'$ such that $x\bar{e}_1'=\bar{e}_2'x\bar{e}_1'$, $y'x\bar{e}_1'=\bar{e}_1'$, $xy'=\bar{e}_2'$. We put $e_1=\bar{e}_1'$, $e_2=1-\bar{e}_1'-\bar{e}_2'$, $e_3=\bar{e}_2'$. Then $\{e_1, e_2, e_3\}$ is a family of mutually orthogonal, mutually equivalent (large) projections, $e_1+e_2+e_3=1$, and the matrix representation of x with respect to this basis has the following properties: $x_{1,1}=0=x_{2,1}$ and $x_{3,1}$ is invertible in $e_1\mathcal{M}e_1$. Hence the last assertion of the corollary follows from Remark 3. Q.E.D.

We note that the matrix representation of the not "thin" element x, obtained in the preceding proof, obviously implies a theorem of BROWN and PEARCY ([1], Theorem 2). So the proof of the commutator theorem may be shortened even in the case of a properly infinite C^*-factor (cf. also [4]).

We have also obtained an extension, to general W^*-algebras, of the theorem of DYER, PORCELLI and ROSENFELD, giving a spectral characterization of the strong radical ([5]).

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