Some integral operators of trace class

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With an Appendix due to JOACHIM WEIDMANN in Frankfurt (GFR)

1. Introduction

In a series of four papers, BIRMAN and SOLOMJAK [18], [19], [20], [21] formulated estimates for the characteristic values of an integral operator. More specifically they considered two measure spaces (X, ϱ) and (Y, τ) and an integral operator mapping $\mathfrak{L}_2(X, \varrho)$ into $\mathfrak{L}_2(Y, \tau)$. Then they formulated estimates for the characteristic values of this operator. Clearly, these estimates imply trace class criteria. According to the summaries of the papers [18], [19], [21] in their main theorems $X=Y=Q^m$, the m-dimensional unit cube and $\varrho(X)\leqq 1$ and $\tau(Y)\leqq 1$. In their third paper [20] they allow X and Y to be unbounded subsets of $\mathscr{R}^m$ provided that either $\varrho(X)\leqq 1$ or $\tau(Y)\leqq 1$. They show how this case can be reduced to their previously treated case. At the same time they give examples of operators which can be reduced to this case.

In this paper the question of trace class criteria is taken up again for the case of $X=Y=\mathscr{R}^+$ and for the case of both measures being the Lebesgue measure.

In Section 2 first we assign a bound to a given integral operator K acting in $\mathfrak{L}_2(\mathscr{R}^+)$. This bound depends on a given set of three positive constants (α, β, γ) and we denote it by $\|K\|(\alpha, \beta, \gamma)$. The first constant α measures, so to speak, the modulus of mean continuity of the kernel $K(\xi, \eta)$ with reference the second variable η. The second constant β, so to speak, measures an additional smallness of this modulus of mean continuity near infinity. The third constant γ measures the smallness of the kernel itself near infinity and for brevity we refer to it as the decay constant. Then in Theorem 2.1 we formulate a trace class criterion with the aid of the bound $\|K\|(\alpha, \beta, \gamma)$. More specifically it is a family of criteria depending on the parameters

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(α, β, γ). The only needed restriction on these parameters is that they satisfy the three inequalities of assumption (2.5) of Theorem 2.1. The first two of these inequalities simply says that α and γ is greater than 1/2. The third inequality involves all of the three parameters (α, β, γ). Roughly speaking it says that for given α the modulus of mean continuity near infinity is small compared to the decay exponent γ. The bigger this decay exponent the less additional smallness of the modulus of mean continuity is required near infinity.

In Section 3 we derive Theorem 2.1 from a Corollary of an abstract Lemma of Gokhberg—Krein [15] which was formulated by Birman—Solomjak [18. a]. The method of our proof differs from theirs inasmuch as the construction of approximating operators does. Specifically with the aid of the set of three positive constants (α, β, γ) of Theorem 2.1 for each positive integer n a partion of $\mathscr{R}^+$ is defined. Then this partion is used to define a subspace of $\mathfrak{L}_2(\mathscr{R}^+)$ and we choose the n-th approximating operator to be the restriction of K to this subspace. In our proof this partition plays the same role as the Birman—Solomjak approximation theorem by piecwise polynomial functions [17] did play in theirs.

The delicate counter-example of the Appendix is due to Weidmann. It illustrates that in assumption (2.5) of Theorem 2.1 one needs a strict inequality.

2. Formulation of the result

Let K be a Hilbert—Schmidt operator acting in $\mathfrak{L}_2(\mathscr{R}^+)$ with kernel $K(\xi, \eta)$. In this section we formulate criteria for K to be in trace class.

To describe these criteria to the operator K and to a given set of three positive constants (α, β, γ) we assign a bound, $\|K\|(\alpha, \beta, \gamma)$. Using the well known formula for the Hilbert—Schmidt norm of an integral operator [4. d] [13. c] and a theorem of Fubini [4. b.] we see that for each bounded interval $\mathscr{I}$ the mean

$$M(\mathscr{I})K(\xi) = \frac{1}{|\mathscr{I}|} \int_{\mathscr{I}} K(\xi, \eta)\, d\eta \tag{2.1}$$

is a square integrable function of the variable ξ. Here, of course, $|\mathscr{I}|$ denotes the lengt of the interval $\mathscr{I}$. The first constant α will measure the smallness of the modulus of mean continuity of the kernel $K(\xi, \eta)$ with reference to the second variable η. More specifically define a preliminary bound by

$$\|K\|(\alpha) = \sup_{\mathscr{I}} \left(\frac{1}{|\mathscr{I}|}\right)^{\frac{2\alpha+1}{2}} \Big(\iint_{\mathscr{R}^+ \times \mathscr{I}} |K(\xi,\eta) - M(\mathscr{I})K(\xi)|^2\, d\xi\, d\eta \Big)^{\frac{1}{2}}. \tag{2.2}$$

Here the supremum is taken over the bounded subintervals of $\mathscr{R}^+$. Incidentally note that for continuous kernels with bounded support the finiteness of this norm is implied by

$$\sup_{\xi}\sup_{\eta_1,\eta_2}\left(\frac{1}{|\eta_2-\eta_1|}\right)^{\alpha}|K(\xi,\eta_2)-K(\xi,\eta_1)|<\infty.$$

This implication is an elementary consequence of the fact that to a given vector with reference to a given subspace the best approximation is the orthogonal projection [4. a] [11. b]. Hence

$$\inf_{\eta_1}\int_{\mathscr{I}}|K(\xi,\eta)-K(\xi,\eta_1)|^2\,d\eta=\int_{\mathscr{I}}|K(\xi,\eta)-M(\mathscr{I})K(\xi)|^2\,d\eta.$$

The second constant β will measure an additional smallness property of the modulus of mean continuity near infinity. More specifically for a given pair of positive constant (α, β) we define a bound by setting

(2.3)$_1$

$$\|K\|_1(\alpha,\beta)=\sup\left(\frac{1}{|\mathscr{I}|}\right)^{\frac{2\alpha+1}{2}}(1+\min\partial\mathscr{I})^{\beta}\Big(\iint_{\mathscr{R}^+\times\mathscr{I}}|K(\xi,\eta)-M(\mathscr{I})K(\xi)|^2\,d\xi\,d\eta\Big)^{\frac{1}{2}}$$

Here, as usual $\partial\mathscr{I}$ denotes the boundary points of $\mathscr{I}$ and supremum is taken over all compact subintervals of $\mathscr{R}^+$. The third constant γ will measure the smallness of the kernel $K(\xi, \eta)$ itself near infinity. For brevity we refer to it as the decay exponent. More specifically with the aid of γ we define a bound by setting

$$\|K\|_2(\gamma)=\sup_{\eta_1}(1+\eta_1)^{\frac{\gamma-1}{2}}\Big(\iint_{(\eta_1,\infty)\times\mathscr{R}^+}|K(\xi,\eta)|^2\,d\xi\,d\eta\Big)^{\frac{1}{2}}. \tag{2.3$_2$}$$

Finally define

$$\|K\|(\alpha,\beta,\gamma)=\max\big(\|K\|_1(\alpha,\beta),\ \|K\|_2(\gamma)\big). \tag{2.4}$$

The theorem that follows formulates a family of trace class criteria for the operator K with the aid of the bound $\|K\|(\alpha, \beta, \gamma)$.

Theorem 2.1. *Let K be a Hilbert—Schmidt operator acting on $\mathfrak{L}_2(\mathscr{R}^+)$. Suppose that to this operator there are three positive constants (α, β, γ) such that*

$$\alpha>1/2,\quad \gamma>1/2,\quad \textit{and}\quad (2\alpha+1-2\beta)<(2\gamma-1)(2\alpha-1) \tag{2.5}$$

and

$$\|K\|(\alpha,\beta,\gamma)<\infty. \tag{2.6}$$

Then this operator is in trace class, specifically

$$K\in\mathfrak{S}_1(\mathfrak{L}_2(\mathscr{R}^+)). \tag{2.7}$$

We shall establish this theorem in the next section. At present let us consider the three inequalities of assumption (2.5) again. The second one, namely that $\gamma > 1/2$ is evident. For, only in this case does the bound $\|K\|_2(\gamma)$ measure smallness at infinity. In fact for $\gamma < 1/2$ this bound is finite for any Hilbert—Schmidt operator. Concerning the first inequality all that is evident is the positivity of α. Nevertheless a straightforward adaptation of the Weidmann example of the Appendix shows that it is possible for a non-trace-class operator to have a finite $\|K\|(\alpha)$ bound with $\alpha = 1/2$. The details of this adaptation were carried out elsewhere [22]. Concerning the third inequality of assumption (2.5) all that we know is that it cannot be sharpened according to the Appendix.

3. Proof of Theorem 2.1

In this section we derive Theorem 2.1 from the Birman—Solomjak Corollary [18. a]. We do not know whether the assumptions of Theorem 2.1 allow one to construct an operator $\tilde{K}$ which is unitarily equivalent to the original operator K and is such that the Birman—Solomjak results of [20] apply to it. We do know, however, that the three constants (α, β, γ) of Theorem 2.1 allow one to construct a partition of $\mathcal{R}^+$. This partition, in turn, allows one to define a sequence of approximating operators satisfying the assumptions of the Birman—Solomjak Corollary [18. a].

Our construction will depend on whether

$$(3.1)_1 \qquad 2\alpha + 1 - 2\beta \leqq 0.$$

or

$$(3.1)_2 \qquad 2\alpha + 1 - 2\beta > 0.$$

In case relation $(3.1)_1$ holds, first we choose a preliminary constant r so that

$$(3.2)_1 \qquad r(2\gamma - 1) > 1.$$

Then set

$$(3.3) \qquad \nu = n^r,$$

and to this r we choose σ so large that

$$(3.4)_1 \qquad 2\alpha - \frac{2\alpha+1}{\sigma} r > 1 \quad \text{and} \quad \sigma - r > 1.$$

In case relation $(3.1)_2$ holds first we choose the preliminary constant r so that

$$(3.2)_2 \qquad r(2\gamma - 1) > 1 \quad \textit{and} \quad 2\alpha - r(2\alpha + 1 - 2\beta) > 1.$$

Then as before define ν be equation (3.3). At present we choose σ so large that

$$(3.4)_2 \qquad \max(2\alpha+1-2\beta),\ \frac{2\alpha+1}{\sigma} = 2\alpha+1-2\beta \quad \text{and} \quad \sigma - r > 1.$$

Note that the two inequalities in $(3.2)_2$ together with relation $(3.1)_2$ are equivalent to

$$\frac{1}{2\gamma-1} < r < \frac{2\alpha-1}{2\alpha+1-2\beta}.$$

Remembering assumption (2.5) and using relation $(3.1)_2$ again we see that this inequality does admit a solution r. Let us emphasize again that the definition of the constants ν, σ depends on whether relation $(3.1)_1$ or $(3.1)_2$ holds. Having defined these constants for each positive integer n we define a function $g_n(\nu, \sigma)$ by

$$(3.5) \qquad g_n(\nu, \sigma)(x) = \nu\left(\frac{x}{n}\right)^{\sigma}.$$

Finally with the aid of this function we define a family of intervals by

$$(3.6) \qquad \mathscr{I}_n(i, \nu, \sigma) = [g_n(\nu, \sigma)(i), g_n(\nu, \sigma)(i+1)), \quad i = 0, 1, 2, \ldots, n-1.$$

According to definitions $(3.5)_{1,2}$, it is no loss of generality to assume that $\sigma > 1$, which implies that this function is strictly increasing. Then clearly this family of intervals defines a partition of the interval $[0, \nu)$.

Next let $c_n(i, \nu, \sigma)$ denote the characteristic function of the interval $\vartheta_n(i, \nu, \sigma)$ and define the subspace $\mathfrak{K}_n(\alpha, \beta, \gamma)$ to be their linear span. Specifically

$$(3.7) \qquad \mathfrak{K}_n(\alpha, \beta, \nu) = \{c_n(i, \nu, \sigma); \quad i = 0, 1, 2, \ldots, n-1\}.$$

Note that this subspace depends on the constants α, β, γ inasmuch as ν and σ do. Clearly

$$(3.8) \qquad \dim \mathfrak{K}_n(\alpha, \beta, \gamma) = n.$$

Let P_n denote the ortho-projector on $\mathfrak{K}_n(\alpha, \beta, \gamma)$ and set

$$(3.9) \qquad K_n = KP_n.$$

Then for each positive integer n this equation defines an operator of rank n. This fact allows us to apply the Birman—Solomjak Corollary [18. a]. According to this corollary for the $(2n+1)$-st characteristic value of the Hilbert—Schmidt operator K, that is for the $(2n+1)$-st eigenvalue of the positive self-adjoint operator $(K^*K)^{1/2}$, we have

$$(3.10) \qquad \mu_{2n+1}(K) \leqq \left(\frac{1}{n}\right)^{\frac{1}{2}} \|K-K_n\|\,(\text{H.S}).$$

Here the second factor denotes the Hilbert—Schmidt norm. Actually this estimate holds for any operator of rank n. The lemma that follows will imply that for the operator K_n of definition (3.9) the right members of estimate (3.10) from a convergent series.

Lemma 3.1. *Suppose that the operator K satisfies the assumptions of Theorem 2.1 and define the operator K_n by equation (3.9). Then there are constants $\delta(\alpha, \beta, \gamma)$ and $\lambda(\alpha, \beta, \gamma)$ such that*

$$\delta(\alpha, \beta, \gamma) > 1/2 \tag{3.11}$$

and for every positive integer n we have

$$\|K-K_n\|(\text{H.S}) \leqq \lambda(\alpha, \beta, \gamma)\left(\frac{1}{n}\right)^{\delta(\alpha, \beta, \gamma)}. \tag{3.12}$$

To establish conclusion (3.11) set

$$\varkappa(\alpha, \beta, \sigma) = \max\left(2\alpha+1-2\beta, \frac{2\alpha+1}{\sigma}\right) \tag{3.13}$$

and

$$\delta(\alpha, \beta, \gamma) = 1/2 \min\{2\alpha-\varkappa(\alpha, \beta, \sigma)r, (2\gamma-1)r\}. \tag{3.14}$$

Here the constants r and σ are defined by equations $(3.2)_{1,2}$ and $(3.4)_{1,2}$. At the same time we see from these definitions that this constant $\delta(\alpha, \beta, \gamma)$ is greater than 1/2. That is to say, conclusion (3.11) holds.

To establish conclusion (3.12) first we introduce a notation for the difference in (3.12) by setting

$$D_n = K-K_n \tag{3.15}$$

Remembering definition (3.9) an elementary argument shows that the kernel of this operator is given by

$$D_n(\xi, \eta) = \begin{cases} K(\xi, \eta)-M(\mathscr{I}_n(i, \nu, \sigma))K(\xi) & \text{for} \quad \eta\in\mathscr{I}_n(i, \nu, \sigma) \\ K(\xi, \eta) & \text{for} \quad \eta\in[\nu, \infty). \end{cases} \tag{3.16}$$

Next we introduce two more operators by setting

$$D_{n,1}(\xi, \eta) = \begin{cases} D_n(\xi, \eta) & \eta\in[0, \nu) \\ 0 & \eta\in[\nu, \infty) \end{cases} \tag*{$(3.17)_1$}$$

and

$$D_{n,2}(\xi, \eta) = \begin{cases} 0 & \eta\in[0, \nu) \\ K(\xi, \eta) & \eta\in[\nu, \infty). \end{cases} \tag*{$(3.17)_2$}$$

Remembering definition (3.15) we see that

$$D_n = D_{n,1}+D_{n,2}. \tag{3.18}$$

To estimate the square of the Hilbert—Schmidt norm of the operator $D_{n,1}$ we need a notation. Specifically for each positive integer n set

$$(3.19)\qquad S_n(\alpha,\beta,g_n(\nu,\sigma)) = \sum_{i=0}^{n-1} \frac{|g_n(\nu,\sigma)(i+1)-g_n(\nu,\sigma)(i)|^{2\alpha+1}}{(1+g_n(\nu,\sigma)(i))^{2\beta}}.$$

Then we claim that

$$(3.20)_1\qquad \|D_{n,1}\|^2(\mathrm{H.S}) \leqq \|K\|_1^2(\alpha,\beta)S_n(\alpha,\beta,g_n(\nu,\sigma)).$$

For definition $(3.17)_1$ together with the partition property of the intervals $\{\mathscr{I}_n(i,\nu,\sigma)\}$ yields

$$(3.21)\qquad \|D_{n,1}\|^2(\mathrm{H.S}) = \sum_{i=0}^{n-1} \iint_{\mathscr{R}^+\times\mathscr{I}_n(i,\nu,\sigma)} |D_n(\xi,\eta)|^2\,d\xi\,d\eta,$$

if we use the well known formula [4. d] [13. c] for the square of the Hilbert—Schmidt norm of an integral-operator. Definitions $(2.3)_1$, (3.7), and relation (3.16) together show that

$$\iint_{\mathscr{R}^+\times\mathscr{I}_n(i,\nu,\sigma)} |D_n(\xi,\eta)|^2\,d\xi\,d\eta \leqq \|K\|_1^2(\alpha,\beta)\frac{|g_n(\nu,\sigma)(i+1)-g_n(\nu,\sigma)(i)|^{2\alpha+1}}{(1+g_n(\nu,\sigma)(i))^{2\beta}}.$$

Inserting this estimate in equality (3.21) and remembering definition (3.19) we obtain the validity of estimate $(3.20)_1$.

In the technical lemma that follows we estimate this sum in terms of n and ν. Actually this is slightly more general than what we need inasmuch as we do not assume that ν is a given function of n.

Lemma 3.2. *For each positive integer n and pair of positive constants ν, σ define the function $g_n(\nu,\sigma)$ by equation* (3.5). *Let α, β be a given pair of positive constants and define the sum $S_n(\alpha,\beta,g_n(\nu,\sigma))$ by equation* (3.19). *Then to each α, β and $\sigma>1$ there is a constant $\gamma(\alpha,\beta,\sigma)$ such that defining the constant $\varkappa(\alpha,\beta,\sigma)$ by equation* (3.13) *for every (ν,n) in $(1,\infty)\times(1,\infty)$ we have*

$$(3.22)\qquad S_n(\alpha,\beta,g_n(\nu,\sigma)) \leqq \gamma(\alpha,\beta,\sigma)\left(\frac{1}{n}\right)^{2\alpha}\nu^{\varkappa(\alpha,\beta,\sigma)}+\left(\frac{\nu}{n^\sigma}\right)^{2\alpha+1}.$$

The assumption $\sigma>1$ together with definition (3.5) shows that the derivative of $g_n(\nu,\sigma)$ is increasing. This, in turn, together with the mean value theorem shows that for $i=1,2,\dots,n-1$,

$$|g_n(\nu,\sigma)(i+1)-g_n(\nu,\sigma)(i)| \leqq 2^\sigma\left(\frac{\nu\sigma}{n^\sigma}\right)(i)^{\sigma-1}.$$

At the same time definition (3.5) shows that for $i=0$,

$$|g_n(\nu,\sigma)(1)-g_n(\nu,\sigma)(0)| = \nu\left(\frac{1}{n}\right)^\sigma.$$

Inserting these inequalities and definition (3.5) in definition (3.19) yields

$$S_n(\alpha,\beta,g_n(v,\sigma)) \leqq \left(2^\sigma \frac{v\sigma}{n^\sigma}\right)^{2\alpha+1} \sum_{j=1}^{n-1} \frac{j^{(\sigma-1)(2\alpha+1)}}{\left(1+v\left(\frac{j}{n}\right)^\sigma\right)^{2\beta}} + \left(\frac{v}{n^\sigma}\right)^{2\alpha+1}. \tag{3.23}$$

To estimate this sum define the function $f_n(\alpha, \beta, v, \sigma)$ by

$$f_n(\alpha, \beta, v, \sigma)(x) = \frac{x^{(\sigma-1)(2\alpha+1)}}{\left(1+v\left(\frac{x}{n}\right)^\sigma\right)^{2\beta}}. \tag{3.24}$$

First we consider the case of

$$\varkappa(\alpha, \beta, \sigma) = 2\alpha+1-2\beta. \tag{3.25$_1$}$$

We claim that this implies that the function $f_n(\alpha, \beta, v, \sigma)$ of definition (3.24) is increasing on the positive real axis. For, suppose that its derivative does vanish at some point m. Then elementary algebra shows that m satisfies the equation

$$[(\sigma-1)(2\alpha+1)-2\beta\sigma]v\left(\frac{m}{n}\right)^\sigma = -(\sigma-1)(2\alpha+1). \tag{3.26}$$

By assumption the right member is strictly negative. It is an elementary consequence of definition (3.13) and relation $(3.25)_1$ that the expression in the bracket is positive, that is

$$(\sigma-1)(2\alpha+1)-2\beta\sigma \geqq 0. \tag{3.27$_1$}$$

Hence there is no point m on the positive axis where this derivative does vanish, and our claim follows. The increasing character of this function shows that

$$\sum_{j=1}^{n-1} \frac{j^{(\sigma-1)(2\alpha+1)}}{\left(1+v\left(\frac{j}{n}\right)^\sigma\right)^{2\beta}} < \int_1^n \frac{x^{(\sigma-1)(2\alpha+1)}}{\left(1+v\left(\frac{x}{n}\right)^\sigma\right)^{2\beta}}\,dx. \tag{3.28}$$

Clearly

$$\int_1^n \frac{x^{(\sigma-1)(2\alpha+1)}}{\left(1+v\left(\frac{x}{n}\right)^\sigma\right)^{2\beta}}\,dx \leqq n^{1+(\sigma-1)(2\alpha+1)} \int_0^1 \frac{y^{(\sigma-1)(2\alpha+1)}}{(1+vy^\sigma)^{2\beta}}\,dy \tag{3.29}$$

and

$$\int_0^1 \frac{y^{(\sigma-1)(2\alpha+1)}}{(1+vy^\sigma)^{2\beta}}\,dy = \left(\frac{1}{v}\right)^{2\beta} \int_0^1 \frac{y^{(\sigma-1)(2\alpha+1)}}{\left(\frac{1}{v}+y^\sigma\right)^{2\beta}}\,dy. \tag{3.30}$$

Remembering relation $(3.27)_1$ we see that the integral on the right is bounded independently of v. In fact

$$\int_0^1 \frac{y^{(\sigma-1)(2\alpha+1)}}{\left(\frac{1}{v}+y^\sigma\right)^{2\beta}}\,dy \leqq 1.$$

This relation together with relations (3.29) and (3.30) inserted in estimate (3.28) yields

$$\sum_{j=1}^{n-1} \frac{j^{(\sigma-1)(2\alpha+1)}}{\left(1+\nu\left(\frac{j}{n}\right)^{\sigma}\right)^{2\beta}} \leqq n^{1+(\sigma-1)(2\alpha+1)}\left(\frac{1}{\nu}\right)^{2\beta}. \tag{3.31}$$

Inserting this estimate, in turn, in estimate (3.23) we arrive at,

$$S_n(\alpha, \beta, g_n(\nu, \sigma)) \leqq (2^{\sigma}\sigma)^{2\alpha+1}\left(\frac{1}{n}\right)^{2\alpha} \nu^{2\alpha+1-2\beta} + \left(\frac{\nu}{n^{\sigma}}\right)^{2\alpha+1}.$$

Hence setting

$$\gamma_1(\alpha, \beta, \sigma) = (2^{\sigma}\sigma)^{2\alpha+1} \tag{3.32$_1$}$$

and remembering definition (3.13) we arrive at the validity of conclusion (3.22) in case relation $(3.25)_1$ holds.

Second we consider the case of

$$\varkappa(\alpha, \beta, \sigma) = \frac{2\alpha+1}{\sigma}. \tag{3.25$_2$}$$

If the two numbers in definition (3.13) are equal then relation $(3.25)_1$ also holds and we have just seen the validity of conclusion (3.22). Accordingly we assume that

$$2\alpha+1-2\beta < \frac{2\alpha+1}{\sigma}.$$

Clearly, this implies that

$$(\sigma-1)(2\alpha+1)-2\beta\sigma < 0, \tag{3.27$_2$}$$

which, in turn, implies that equation (3.26) does admit a positive solution m. That is to say the derivative of the function $f_n(\alpha, \beta, \nu, \sigma)$ does vanish at m. Hence this function is increasing on the interval $[0, m]$ and decreasing on the interval $[m, \infty)$. This fact together with definition (3.24) shows that

$$\sum_{j=1}^{n-1} \frac{j^{(\sigma-1)(2\alpha+1)}}{\left(1+\nu\left(\frac{j}{n}\right)^{\sigma}\right)^{2\beta}} \leqq \int_1^n \frac{x^{(\sigma-1)(2\alpha+1)}}{\left(1+\nu\left(\frac{x}{n}\right)^{\sigma}\right)^{2\beta}} dx + \frac{m^{(\sigma-1)(2\alpha+1)}}{\left(1+\nu\left(\frac{m}{n}\right)^{\sigma}\right)^{2\beta}}. \tag{3.33}$$

It is not difficult to estimate this integral. In fact we claim that relation $(3.27)_2$ implies

$$\int_1^n \frac{x^{(\sigma-1)(2\alpha+1)}}{\left(1+\nu\left(\frac{x}{n}\right)^{\sigma}\right)^{2\beta}} dx \leqq \left(\frac{n^{\sigma}}{\nu}\right)^{(2\alpha+1)-\frac{2\alpha}{\sigma}} (2\nu)^{\frac{1}{\sigma}}. \tag{3.34}$$

For, setting

$$\nu\left(\frac{x}{n}\right)^{\sigma}=t,$$

an elementary change of variables, yields

$$\int_1^n \frac{x^{(\sigma-1)(2\alpha+1)}}{\left(1+\nu\left(\frac{x}{n}\right)^{\sigma}\right)^{2\beta}}dx \leqq \frac{1}{\sigma}\left(\frac{n^{\sigma}}{\nu}\right)^{(2\alpha+1)-\frac{2\alpha}{\sigma}} \int_0^{\nu} \frac{t^{\frac{\sigma-1}{\sigma}2\alpha}}{(1+t)^{2\beta}}\,dt.$$

At the same time, we see from relation $(3.27)_2$ that

$$\frac{\sigma-1}{\sigma}2\alpha-2\beta<\frac{1}{\sigma}-1.$$

This shows that

$$\int_0^{\nu}(1+t)^{\frac{\sigma-1}{\sigma}2\alpha-2\beta}\,dt \leqq (2\nu)^{\frac{1}{\sigma}}.$$

if we remember that by assumption $\nu>1$. Inserting this estimate in the previous one we obtain the validity of estimate (3.34).

It is not difficult to estimate the second term in (3.33) either. To do this recall that the positive number m was defined by equation (3.26). This equation together with relation $(3.27)_2$ shows that setting

$$\tilde{\gamma}(\alpha,\beta,\sigma)=\frac{-(\sigma-1)(2\alpha+1)}{(\sigma-1)(2\alpha+1)-2\beta\sigma},$$

we have

$$\nu\left(\frac{m}{n}\right)^{\sigma}=\tilde{\gamma}(\alpha,\beta,\sigma)$$

Then elementary algebra shows that equation (3.26) implies

$$\text{(3.35)}\qquad \frac{m^{(\sigma-1)(2\alpha+1)}}{\left(1+\nu\left(\frac{m}{n}\right)^{\sigma}\right)^{2\beta}} \leqq \left(\tilde{\gamma}(\alpha,\beta,\sigma)\right)^{\frac{(\sigma-1)(2\alpha+1)}{\sigma}-2\beta}\left(\frac{n^{\sigma}}{\nu}\right)^{(2\alpha+1)-\frac{2\alpha+1}{\sigma}}$$

Inserting estimates (3.35) and (3.34) in estimate (3.33) we obtain

$$\text{(3.36)}\qquad \sum_{j=1}^{n-1}\frac{j^{(\sigma-1)(2\alpha+1)}}{\left(1+\nu\left(\frac{j}{n}\right)^{\sigma}\right)^{2\beta}} \leqq \left(\frac{n^{\sigma}}{\nu}\right)^{(2\alpha+1)-\frac{2\alpha}{\sigma}}(2\nu)^{\frac{1}{\sigma}}+$$
$$+\left(\tilde{\gamma}(\alpha,\beta,\sigma)\right)^{\frac{(\sigma-1)(2\alpha+1)}{\sigma}-2\beta}\left(\frac{n^{\sigma}}{\nu}\right)^{(2\alpha+1)-\frac{2\alpha+1}{\sigma}}.$$

Inserting this estimate (3.36), in turn, in estimate (3.23) and setting

$$(3.32)_2 \qquad \gamma_2(\alpha,\beta,\sigma) = (2^\sigma\sigma)^{2\alpha+1}\left[2+(\tilde{\gamma}(\alpha,\beta,\sigma))^{\frac{(\sigma-1)(2\alpha+1)}{\sigma}-2\beta}\right],$$

we arrive at the validity of conclusion (3.22) in case relation $(3.25)_2$ holds. This completes the proof of Lemma 3.2, if we remember that according to definition (3.13) either relation $(3.25)_1$ or relation $(3.25)_2$ holds.

Having established Lemma 3.2 we can easily derive conclusion (3.12) of Lemma 3.1 from it. For, insertion of conclusion (3.22) in estimate $(3.20)_1$ yields

$$(3.37)_1 \qquad \|D_{n,1}\|^2(\mathrm{H.S}) \leqq \|K\|_1^2(\alpha,\beta)\gamma^2(\alpha,\beta,\sigma)\left(\frac{1}{n}\right)^{2\alpha-\varkappa(\alpha,\beta,\sigma)r}+\left(\frac{1}{n}\right)^{(\sigma-r)(2\alpha+1)}$$

if we remember definition (3.3). According to relation $(3.17)_1$

$$\|D_{n,2}\|^2(\mathrm{H.S}) = \iint_{(\nu,\infty)\times\mathcal{R}^+} |K(\xi,\eta)|^2\,d\xi\,d\eta.$$

This relation together with definitions $(2.3)_2$ and (3.3) yields

$$(3.37)_2 \qquad \|D_{n,2}\|^2(\mathrm{H.S}) \leqq \|K\|_2^2(\gamma)\left(\frac{1}{n}\right)^{(2\gamma-1)r}.$$

Thus setting

$$\lambda(\alpha,\beta,\gamma) = [\|K\|_1^2(\alpha,\beta)\left(1+\gamma(\alpha,\beta,\sigma)+\|K\|_2^2(\gamma)\right]^{1/2}$$

and remembering definition (3.14) we arrive at the validity of conclusion (3.12) of Lemma 3.1.

Finally we can easily derive Theorem 3.1 from Lemma 2.1. For, insertion of conclusion (3.12) in the Birman—Solomjak Corollary (3.10) yields

$$\mu_{2n+1}(K) \leqq \lambda(\alpha,\beta,\gamma)\left(\frac{1}{n}\right)^{1/2+\delta(\alpha,\beta,\gamma)}$$

According to conclusion (3.11) of Lemma 3.1 the right members form a convergent series. Hence

$$(3.38) \qquad \sum_{n=1}^{\infty} \mu_{2n+1}(K) < \infty.$$

Since the characteristic values were ordered in decreasing order

$$\sum_{n=1}^{\infty} \mu_n(K) \leqq \mu_1(K)+\mu_2(K)+2\sum_{n=1}^{\infty}\mu_{2n+1}(K).$$

Hence inserting estimate (3.38) in this inequality we arrive at the validity of conclusion (2.7), if we remember the definition of trace class [13. d]. This completes the proof of Theorem 2.1.

Appendix

A Hilbert—Schmidt operator with $||K||\,(1, 1, 1) < \infty$ which is not in trace class

By JOACHIM WEIDMANN

Before constructing such an operator note that for $\alpha=1$, $\beta=1$, $\gamma=1$ we have

$$2\alpha+1-2\beta = (2\gamma-1)(2\alpha-1).$$

In other words for these constants the third inequality in assumption (2.5) of Theorem 2.1 is replaced by an equality.

To construct such an operator we first define two ortho-normal sets of functions in $\mathfrak{L}_2(\mathfrak{R}^+)$ by setting

$$a_i(\xi) = \begin{cases} \sqrt{2}\sin(i\pi\xi) & \xi\in[0, 1) \\ 0 & \xi\in[1, \infty) \end{cases} \tag{A-1}$$

and

$$b_i(\eta) = \begin{cases} \sqrt{2}\sin(\pi\eta) & \eta\in[i, i+1] \\ 0 & \eta\notin[i, i+1]. \end{cases} \tag{A-2}$$

It is an immediate consequence of this ortho-normality that setting

$$K(\xi, \eta) = \sum_{i=1}^{\infty}\frac{1}{i+2}a_i(\xi)b_i(\eta), \tag{A-3}$$

we have

$$\int_0^{\infty}\int_0^{\infty} |K(\xi, \eta)|^2\, d\xi\, d\eta = \sum_{i=1}^{\infty}\left(\frac{1}{i+2}\right)^2 < \infty. \tag{A-4}$$

That is to say this kernel defines a Hilbert—Schmidt operator K. At the same time it follows that

$$(K^*K)^{1/2}(\xi, \eta) = \sum_{i=1}^{\infty}\frac{1}{i+2}a_i(\xi)a_i(\eta), \tag{A-5}$$

and hence

$$\operatorname{tr}[(K^*K)^{1/2}] = \sum_{i=1}^{\infty}\frac{1}{i+2} = \infty. \tag{A-6}$$

In other words the operator K of definition (A-3) is not in trace class.

Next we maintain that its kernel is Hölder continuous. More specifically we maintain that

$$\|K\|(1, 1, 1) < \infty. \tag{A-7}$$

To establish this estimate first we claim that

$$\sup_{\eta_1}(1+\eta_1)^{1/2}\Big(\int_{\eta_1}^{\infty}\int_0^{\infty} K^2(\xi, \eta)\, d\xi\, d\eta\Big)^{1/2} < \infty. \tag{A-8}$$

For, in view of the ortho-normality of the system $\{a_i\}$ at each point η definition (A-3) yields

$$\int_0^\infty K^2(\xi, \eta)\,d\xi = \sum_{i=1}^\infty \left(\frac{1}{i+1}\right)^2 b_i^2(\eta).$$

At the same time we see from definition (A-2) that

$$i+1 < \eta_1 \quad \text{implies} \quad \int_{\eta_1}^\infty b_i^2(\eta)\,d\eta = 0.$$

Hence

$$\int_{\eta_1}^\infty \int_0^\infty K^2(\xi, \eta)\,d\xi\,d\eta \leqq \sum_{i=\eta_1}^{\infty} \frac{1}{(i+1)^2} \leqq \frac{1}{1+\eta_1},$$

and the validity of (A-8) follows. Second we claim that for every (ξ, η_1) and (ξ, η_2) we have

(A-9) $$|K(\xi, \eta_1) - K(\xi, \eta_2)| \leqq 2\pi \frac{|\eta_2 - \eta_1|}{(1+\min(\eta_1, \eta_2))}.$$

For, in case η_1 and η_2 are in the same interval, say

(A-10) $$\eta_1 \in [m, m+1) \quad \text{and} \quad \eta_2 \in [m, m+1),$$

definition (A-3) yields

(A-11) $$|K(\xi, \eta_2) - K(\xi, \eta_2)| = \left|\frac{a_m(\xi)}{m+2}\left(b_m(\eta_2) - b_m(\eta_1)\right)\right|.$$

Remembering definition (A-2) we see from the mean value theorem that

(A-12) $$|b_m(\eta_2) - b_m(\eta_1)| \leqq \sqrt{2\pi}\,|\eta_2 - \eta_1|.$$

Definition (A-1) together with assumption (A-10) yields

(A-13) $$\left|\left(\frac{1}{m+2}\right) a_m(\xi)\right| \leqq \frac{\sqrt{2}}{1+\eta}.$$

Hence in this case relation (A-11) together with estimates (A-12) and (A-13) yields the validity of estimate (A-9). In the general case let the integers $m_{1,2}$ be defined by

(A-14) $$m_1 \leqq \eta_1 \leqq m_1 + 1 \leqq m_2 \leqq \eta_2 < m_2 + 1.$$

Then definition (A-3) yields

(A-15) $$|K(\xi, \eta_2) - K(\xi, \eta_1)| \leqq \sum_{i=1}^{2} \left|\frac{a_{m_i}(\xi)}{m_i + 2} b_{m_i}(\eta_i)\right|,$$

if we use the triangle inequality.

Since

$$\sin((m_1+1)\pi) = 0,$$

the mean value theorem implies for $i=1, 2$

$$\left|b_{m_i}(\eta_i)-\sin\left((m_1+1)\pi\right)\right| < \sqrt{2\pi}\,|\eta_i-(m_1+1)\eta|.$$

Assumption (A-14) clearly implies that

$$|\eta_1-(m_1+1)|+|\eta_2-(m_1+1)\eta| = \eta_2-\eta_1.$$

At the same time, similarly to (A-13) we have for $i=1, 2$,

$$\left|\frac{a_{m_i}(\xi)}{m_i+2}\right| \leqq \frac{\sqrt{2}}{1+\min(\eta_1, \eta_2)}.$$

Inserting these three relations in estimate (A-15) we arrive at the validity of estimate (A-9). Remembering that for ξ in $[1, \infty)$ the kernel $K(\xi, \eta)$ vanishes, we see that estimate (A-9) implies

$$\sup\,(1+\min(\eta_1, \eta_2))\left(\frac{1}{|\eta_2-\eta_1|}\right)^{3/2}\Big(\int_{\eta_1}^{\eta_2}\int_0^{\infty}|K(\xi, \eta)-K(\xi, \eta_2)|^2\,d\xi\,d\eta\Big)^{1/2} < \infty.$$

Finally combining estimates (A-8) and (A-16) we arrive at the validity of estimate (A-7).

References

[1] T. Lalesco, *Introduction à la théorie des équations intégrales,* Hermann (Paris, 1912). See pp. 86—89.

[2] H. Weyl, Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen, *Math. Ann.*, **11** (1912), 441—479.

[3] E. Hille, and J. D. Tamarkin, On the characteristic values of linear integral equations, *Acta Math.*, **57** (1931), 1—76.

[4] F. Riesz and B. Sz.-Nagy, *Leçons d'analyse fonctionnelle,* Akadémiai Kiadó (Budapest, 1952), English translation: *Functional Analysis,* Frederick Ungar Publishing Co. (New York, 1955). a. Section 33; b. Section 40; c. Section 95; d. Section 97.

[5] M. G. Krein, On the trace formula in perturbation theory. *Mat. Sbor.*, **33** (1953), 597—626. (Russian).

[6] Forrest W. Stinespring, A sufficient condition for an integral operator to have a trace, *J. reine angew. Math.*, **200** (1958), 200—207.

[7] R. B. Lavine, *The Weyl Transform — Fourier analysis for operators in L_p-spaces,* Thesis, M.I.T., 1965.

[8] J. Weidmann, Integraloperatoren der Spurklasse, *Math. Ann.*, **163** (1966), 340—345.

[9] C. A. McCarthy, „C_p", *Israel J. Math.*, **5** (1967), 249—271.

[10] R. H. Lorentz, *Gentleness versus trace class.* Thesis, University of Minnesota, 1969.

[11] R. Courant, and D. Hilbert, *Methods of Mathematical Physics,* Vol. 1. Interscience, John Wiley (New York, 1953). a. Section I.4; b. Subsection II.1.3.

[12] K. O. Friedrichs, *Perturbation of Spectra in Hilbert Space.* Amer. Math. Soc. (Providence, 1965). See Appandix 6.

[13] N. Dunford, and J. T. Schwartz, *Linear Operators,* Part II, Interscience, John Wiley (New York, 1963). a. Theorem X.4.3; b. Definition XI.6.1; c. Exercise XI.8.44; d. Definition XI.9.1; e. Section XI.9.32.

[14] T. Kato, *Perturbation Theory for Linear Operators.* Springer Verlag (1966). See Section X.4.

[15] I. C. Gohberg, and M. A. Krein, *Introduction to the theory of nonselfadjoint operators.* Volume Eighteen, Translations of Mathematical Monographs, Amer. Math. Soc. (Providence, Rhode Island, 1969). See Lemma III.6.1.

[16] J. Dixmier, *Les algèbres d'opérateurs dans l'espace Hilbertien.* Gauthier-Villars (Paris, 1969) (Second Edition).

[17] M. Sh. Birman and M. Z. Solomjak, Piecewise polynomial approximations of functions of classes W_p^α, *Mat. Sbornik,* (N. S.) **73** (**115**) (1967), 331—335 (Russian), *Math. Rev.,* **36,** No. 576.

[18] M. Sh. Birman and M. Z. Solomjak, On estimates for singular values of integral operators. I, *Vestnik Leningrad Univ.,* (22) No. **7**, (1967), 43—53 (Russian), *Math. Rev.,* **35,** No. 7173. a. Corollary after Lemma 1.

[19] M. Sh. Birman and M. Z. Solomjak, On estimates for singular values of integral operators. II, *Vestnik Leningrad Univ.,* (22) No. **13**, (1967), 21—28 (Russian), *Math. Rev.,* **36,** No. 739.

[20] M. Sh. Birman and M. Z. Solomjak, On estimates for the singular values of integral operators. III. Operators in unbounded domains, *Vestnik Leningrad Univ.,* (24) No. **1** (1969), 35—48 (Russian), *Math. Rev.* **39**, No. 7468.

[21] M. Z. Solomjak, On estimates for singular values of integral operators. IV, *Vestnik Leningrad Univ.,* No. **1** (1970), 76—87 (Russian), Math. Rev., **41**, No. 9066.

[22] R. H. Lorentz and P. A. Rejtő, *Some integral operators of trace class.* Batelle Institute, Advanced Studies Center, Geneva, Switzerland. Technical Report 50, 1971. See Appendix I.

[23] M. Sh. Birman and M. Z. Solomjak, Remarks on the nuclearity of integoperators and the boundedness of pseudodifferential operators. *Izv. Vysš. Učebn. Zaved. Matematika 1969,* no 9 (**88**) 11—17. (Russian), *Math. Rev.,* **40**, No. 7877.

[24] Ph. Martin and B. Misra, On trace-class operators of scattering theory and the asymptotic behaviour of scattering cross section at high energy. *J. Math. Phys.,* **14** (1973), 997—1005.

[25] Eugène B. Fabes, Walter Littman and Nestor M. Riviere, Transformers of pseudodifferential operators, *Notices Amer. Math. Soc.,* **21** (1974).

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