On fractional powers of operators in Hilbert space

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0. The primary concern of this note is to give conditions (Theorem 1) such that if A and B are each self adjoint operators with positive lower bounds and $A+B$ is self adjoint, then for $0 \leqq \tau \leqq 1$, the domain $D((A+B)^\tau)$ equals $D(A^\tau) \cap D(B^\tau)$. A theorem of LIONS and MAGENES [19] on interpolation of intersections is then obtained as a corollary. It is then verified that for a large class of Schrödinger operators $-\Delta+q(x)$ on R^n, $\Delta=$Laplacian, q real valued, the conditions are satisfied so that Theorem 1 is applicable if $D(-\Delta+q(x))=D(-\Delta)\cap D(q(x))$ in the operator theoretic sense.

In addition a new sufficient condition (Theorem 2) for the equality of $D(C^{1/2})$ and $D(C^{*1/2})$, where C is a regularly accretive operator, is given. This condition is shown to be applicable if C arises as an elliptic partial differential operator with homogeneous Dirichlet boundary conditions over certain (possibly unbounded) domains admitting corners, the Lipschitzian graph domains.

1. Let H be a complex Hilbert space with norm $|u|$ and inner product (u, v). Further let V_a (resp. V_b) be a complex Hilbert space with $V_a \underset{c}{\subseteq} H$ (resp. $V_b \underset{c}{\subseteq} H$), i.e. V_a is a vector subspace of H and the injection of V_a into H is continuous. Also assume that V_a, V_b, and $V_a \cap V_b$ are dense in H and denote the inner product in V_a (resp. V_b) by $a(u, v)$ (resp. $b(u, v)$). To the inner product $a(u, v)$ there corresponds a linear operator A in H, *the operator in H associated with* $a(u, v)$, defined on

$$D(A) = \{u \in V_a : v \to a(u, v) \text{ is continuous on } V_a \text{ in the topology induced by } H\}$$

by

$$(Au, v) = a(u, v) \quad \text{for all} \quad v \in V_a.$$

A is a positive definite self adjoint operator in H and $D(A)$ is dense in V_a. For τ positive, denote by A^τ the positive τth power of A as defined by use of the spectral

* The research for this paper was supported in part by the National Science Foundation.

theorem; A^τ is a positive definite, self adjoint operator in H. Furthermore, $D(A^{1/2})$ is V_a and $a(u, v)=(A^{1/2}u, A^{1/2}v)$ for all $u, v\in V_a$.

For $0\leqq\tau\leqq 1$, the τth *interpolation space by quadratic interpolation* between V_a and H, $[V_a, H]_\tau$, is defined as the Hilbert space

$$[V_a, H]_\tau = D(A^{\tau/2})$$

with inner product $(A^{\tau/2}u, A^{\tau/2}v)$. Further for $\tau\in[0, \infty)$ let $[V_a, H]_\tau$ be the Hilbert space $D(A^{\tau/2})$ with inner product $(A^{\tau/2}u, A^{\tau/2}v)$.

Let B be the operator in H associated with $b(u, v)$, i.e.

$$(Bu, v) = b(u, v), \quad u\in D(B), \quad Bu\in H, \quad v\in V_b,$$

and for $\tau\in[0, \infty)$ denote by $[V_b, H]_\tau$ the Hilbert space $D(B^{\tau/2})$ with inner product $(B^{\tau/2}u, B^{\tau/2}v)$. Now $V_a\cap V_b$, provided with the inner product $a(u, v)+b(u, v)$, is a Hilbert space and, since $V_a\cap V_b$ is dense in H, we may let Σ be the operator in H associated with $a(u, v)+b(u, v)$, i.e.

$$(\Sigma u, v) = a(u, v)+b(u, v), \quad u\in D(\Sigma),$$
$$\Sigma u\in H, \quad v\in V_a\cap V_b.$$

Then for $\tau\in[0, \infty)$ let $[V_a\cap V_b, H]_\tau$ be the Hilbert space $D(\Sigma^{\tau/2})$ with inner product $(\Sigma^{\tau/2}u, \Sigma^{\tau/2}v)$. We wish to obtain relationships between the Hilbert spaces $[V_a\cap V_b, H]_\tau$ and $[V_a, H]_\tau\cap[V_b, H]_\tau$ (with the inner product $(A^{\tau/2}u, A^{\tau/2}v)+(B^{\tau/2}u, B^{\tau/2}v)$), without assuming that $A^{1/2}$ and $B^{1/2}$ commute as in [19], p. 95.

Proposition 1. *For each* $\tau\in[0, 1]$,

$$[V_a\cap V_b, H]_\tau \underset{c}{\subseteq} [V_a, H]_\tau\cap[V_b, H]_\tau,$$

and, if $\alpha, \beta\geqq 0$ *and* $\alpha+\beta = 1$,

$$\alpha|A^{\tau/2}u|+\beta|B^{\tau/2}u| \leqq |\Sigma^{\tau/2}u| \quad \textit{for all} \quad u\in[V_a\cap V_b, H]_\tau.$$

Proof. Obviously the identity mapping is continuous from $V_a\cap V_b$ into V_a with bound $\leqq 1$, continuous from $V_a\cap V_b$ into V_b with bound $\leqq 1$, and continuous from H into H with bound 1. The proposition is thus a trivial consequence of the quadratic interpolation theorem of Lions [16], pp. 431—432 (cf. also Adams, Aronszajn and Hanna [1], App. I).

Observe that $A+B$ is essentially self adjoint if and only if $D(A+B)=D(A)\cap D(B)$ is dense in $D(\Sigma)$, i.e. if and only if $[V_a, H]_2\cap[V_b, H]_2$ is dense in $[V_a\cap V_b, H]_2$. Further if $A+B$ is essentially self adjoint, then the closure of $A+B$ is Σ.

Proposition 2. *If $A+B$ is essentially self adjoint, then for each $\tau\in[1, 2]$ such that $D(A)\cap D(B)$ is dense in $[V_a, H]_\tau\cap[V_b, H]_\tau$,*

$$[V_a, H]_\tau\cap[V_b, H]_\tau \underset{c}{\subseteq} [V_a\cap V_b, H]_\tau,$$

and

(1) $$|\Sigma^{\tau/2}u| \leqq |A^{\tau/2}u|+|B^{\tau/2}u| \quad \textit{for all} \quad u\in[V_a, H]_\tau\cap[V_b, H]_\tau.$$

Proof. Let $u\in D(A)\cap D(B)$. Then since $D(\Sigma)$ is dense in $D(\Sigma^\theta)$ for all $\theta<1$,

$$|\Sigma^{\tau/2}u| = \sup\{|(\Sigma^{\tau/2}u, \Sigma^{1-(\tau/2)}v)| : v\in D(A)\cap D(B) \text{ and } |\Sigma^{1-(\tau/2)}v| = 1\} =$$
$$= \sup\{|(A^{\tau/2}u, A^{1-(\tau/2)}v)+(B^{\tau/2}u, B^{1-(\tau/2)}v| : v\in D(A)\cap D(B) \text{ and } |\Sigma^{1-(\tau/2)}v| = 1\} \leqq$$
$$\leqq \sup\{|(A^{\tau/2}u, A^{1-(\tau/2)}v)| : v\in D(A)\cap D(B) \text{ and } |\Sigma^{1-(\tau/2)}v| = 1\}+$$
$$+\sup\{|(B^{\tau/2}u, B^{1-(\tau/2)}v)| : v\in D(A)\cap D(B) \text{ and } |\Sigma^{1-(\tau/2)}v| = 1\}.$$

Since $2-\tau\in[0, 1]$ it now follows from Proposition 1 that

$$|\Sigma^{\tau/2}u| \leqq \sup\{|(A^{\tau/2}u, A^{1-(\tau/2)}v)| : v\in D(A) \text{ and } |A^{1-(\tau/2)}v| = 1\}+$$
$$+\sup\{|(B^{\tau/2}u, B^{1-(\tau/2)}v)| : v\in D(B) \text{ and } |B^{1-(\tau/2)}v| = 1\} = |A^{\tau/2}u|+|B^{\tau/2}u|.$$

Thus (1) holds for all u in the closure of $D(A)\cap D(B)$ in $[V_a, H]_\tau\cap[V_b, H]_\tau$. The proposition follows.

Observe that $A+B$ is self adjoint if and only if $\Sigma=A+B$ and when this is the case the norms $|\Sigma u|=|(A+B)u|$ and $(|Au|^2+|Bu|^2)^{1/2}$ are equivalent on $D(A)\cap D(B)$ (by the closed graph theorem). In this case $A+B$ is also a topological isomorphism of $D(A)\cap D(B)$ onto H.

Theorem 1. *If $A+B$ is self adjoint, then for each $\tau\in[0, 2]$,*

$$[V_a\cap V_b, H]_\tau \underset{c}{\subseteq} [V_a, H]_\tau\cap[V_b, H]_\tau.$$

Moreover, for each $\tau\in[0, 2]$ such that $D(A)\cap D(B)$ is dense in $[V_a, H]_\tau\cap[V_b, H]_\tau$,

$$[V_a\cap V_b, H]_\tau = [V_a, H]_\tau\cap[V_b, H]_\tau,$$

with equivalent norms.

Proof. The first assertion is obtained by the method of proof of Proposition 1, and the second assertion via the proof of Proposition 2.

Corollary 1. ([19], p. 95) *If H is separable and $A^{1/2}$ and $B^{1/2}$ commute, then for each $\tau\in[0, 2]$,*

$$[V_a\cap V_b, H]_\tau = [V_a, H]_\tau\cap[V_b, H]_\tau$$

with equivalent norms.

Proof. By simultaneous diagonalization of A and B (cf. Dixmier [6], p. 217) it follows in much the same fashion as in the proof of Théorème 13.1, p. 95, [19], that the hypotheses of Theorem 1 are satisfied.

2. In this section we wish to illustrate how the previous results apply to characterization of the domains of fractional powers of Schrödinger operators $-\Delta u+q(x)u$,

$x \in R^n$, Δ=Laplacian, q real and $\geqq 2\delta > 0$. We shall use the theory of Bessel potentials (cf. ARONSZAJN [3], ARONSZAJN and SMITH [5], ADAMS, ARONSZAJN and SMITH [2]).

The Bessel kernel of order $\alpha > 0$ on R^n is the function given by

$$G_\alpha(x) = G_\alpha^{(n)}(x) = \frac{1}{2^{(n+\alpha-2)/2}\pi^{n/2}\Gamma(\alpha/2)} K_{(n-\alpha)/2}(|x|)\,|x|^{(\alpha-n)/2}$$

where K_ν is the modified Bessel function of the 3rd kind. For $0<\alpha<1$, let

$$C(n, \alpha) = \frac{2^{-2\alpha+1}\pi^{(n+2)/2}}{\Gamma(\alpha+1)\Gamma(\alpha+(n/2))\sin \pi\alpha}.$$

Further let D be a domain in R^n and let u be a complex valued function in $C^\infty(D)$. The standard α-norm over D, $|u|_{\alpha, D}$, is defined as follows,

$$|u|_{0,D}^2 = \int_D |u(x)|^2\,dx,$$

and for $0 < \alpha < 1$,

$$|u|_{\alpha,D}^2 = |u|_{0,D}^2 + \frac{1}{C(n,\alpha)G_{2n+2\alpha}(0)} \int_D\int_D \frac{G_{2n+2\alpha}(x-y)}{|x-y|^{n+2\alpha}} |u(x)-u(y)|^2\,dx\,dy.$$

For arbitrary $\alpha \geqq 0$, let $m=[\alpha]$ be the greatest integer $\leqq \alpha$ and let $\beta=\alpha-m$. Then

$$|u|_{\alpha,D}^2 = \sum_{k=0}^{m} \binom{m}{k} \sum_{|i| \leqq k} |D_i u|_{\beta,D}^2.$$

The space $\check{P}^\alpha(D)$ is the perfect functional completion in the sense of ARONSZAJN and SMITH [4] of the functions in $C^\infty(D)$ for which $|u|_{\alpha,D}$ is finite. For $D=R^n$, $\check{P}^\alpha(D)$ is denoted simply by P^α and $|u|_{\alpha,R^n}$ by $\|u\|_\alpha$. $P^\alpha(D)$ is defined as the space of all restrictions to D of functions in P^α with the norm

$$\|u\|_{\alpha,D} = \inf \|\tilde{u}\|_\alpha$$

with the infimum taken over all $\tilde{u} \in P^\alpha$ such that $\tilde{u}=u$ except on a subset of D of 2α-capacity zero. For all domains D to be considered in the present work, $\check{P}^\alpha(D)= =P^\alpha(D)$ with equivalent norms (cf. [2] or [3]). It should be noted that for such domains D, $\check{P}^\alpha(D)$ is the class of corrections (cf. [2], § 0) of functions in the class $W^{\alpha,2}(D)$ (cf. LIONS and MAGENES [18], § 2). Finally recall that $C_0^\infty(R^n)$ is dense in P^α.

Now for $u, v \in P^1$, let

$$a(u, v) = \sum_{|i|=1} \int_{R^n} D_i u\,\overline{D_i v}\,dx + \delta \int_{R^n} u\bar{v}\,dx$$

where $\delta > 0$, and define V_a as the space P^1 with $a(u, v)$ as inner product. Letting $H=L^2(R^n)=P^0$ with the usual inner product, it follows by use of Fourier transforms that the operator A, defined by $a(u, v)=(Au, v)$ is given by $-\Delta u+\delta u$ for $u \in D(A)=P^2$

with an equivalent norm, and that for $0 \leqq \tau \leqq 2$, $D(A^{\tau/2}) = P^\tau$ with an equivalent norm.

Let $q \in L^1_{\text{loc}}(R^n)$ be a real valued function with $q(x) \geqq 2\delta$ a.e. For $u \in L^2(R^n)$, let

$$b(u, u) = \int_{R^n} q(x)|u|^2\,dx - \delta \int_{R^n} |u|^2\,dx$$

and define V_b as the space of all $u \in L^2(R^n)$ such that $b(u, u) < \infty$, with the corresponding inner product $b(u, v)$. Then the operator B, defined by $b(u, v) = (Bu, v)$ is given by $qu - \delta u$ for

$$u \in D(B) = \left\{u \in L^2(R^n) : \int_{R^n} q^2|u|^2\,dx < \infty\right\}$$

and, for $0 \leqq \tau \leqq 2$,

$$D(B^{\tau/2}) = \left\{u \in L^2(R^n) : \int_{R^n} q^\tau|u|^2\,dx < \infty\right\}.$$

Now if q also satisfies the condition that

$$M_{q^2}(x) = \int_{|x-y| \leqq 1} |x-y|^{2-n-\varrho}|q(y)|^2\,dy$$

is locally bounded for some constant $\varrho > 0$, it follows as in KATO [15], pp. 349—351, that each $u \in D(A) \cap D(B)$ can be "mollified", producing a sequence $\{u_n\} \subset C_0^\infty(R^n)$ converging to u in the intersection norm. Then, since the mollifying operation is linear, it follows by interpolation between $D(A)$ and H and between $D(B)$ and H separately, that for each $\tau \in [0, 2]$ and $u \in [V_a, H]_\tau \cap [V_b, H]_\tau$ the mollifiers $\{u_n\} \subset C_0^\infty(R^n)$ converge to u in $[V_a, H]_\tau \cap [V_b, H]_\tau$. Thus $D(A) \cap D(B)$ is dense in $[V_a, H]_\tau \cap [V_b, H]_\tau$ for all $\tau \in [0, 2]$.

Hence for $q \in L^1_{\text{loc}}(R^n)$ such that $M_{q^2}(x)$ is locally bounded, the technical condition, "$D(A) \cap D(B)$ is dense in $[V_a, H]_\tau \cap [V_b, H]_\tau$", is always satisfied. To apply Proposition 2 one may then use criteria for essential self adjointness of $A+B$ to be found e.g. in HELLWIG [9], IKEBE and KATO [10], or JÖRGENS [11]. Conditions on q yielding self adjointness of $A+B$ have been given by TRIEBEL [23], § 6.

3. Let V_a, H be as in Section 1 and let $u, v \to c(u, v)$ be a continuous sesquilinear form on V_a. Further assume that there is a $\gamma > 0$ such that

$$\operatorname{Re} c(v, v) \geqq \gamma a(v, v) \quad \text{for all} \quad v \in V_a.$$

As previously, let C be the operator in H associated with $c(u, v)$, i.e. $(Cu, v) = c(u, v)$ for all $v \in V_a$ with $D(C) = \{u \in V_a : v \to c(u, v)$ is continuous on V_a in the topology induced by $H\}$. Then C is a closed densely defined operator whose domain is also dense in V_a. The adjoint form $c^*(u, v)$, is defined by

$$c^*(u, v) = \overline{c(v, u)}, \quad u, v \in V_a,$$

and if C^* is the operator in H associated with $c^*(u, v)$, i.e., $(C^*u, v)=c^*(u, v)$, $u\in D(C^*)$, $C^*u\in H$, $v\in V_a$, then C^* is the adjoint of C. C and C^* are *regularly accretive* operators in the terminology of KATO [12]. (Kato assumes only that $\mathrm{Re}\, c(v, v)+ +\lambda|v|^2\geqq\gamma a(v, v)$ but replacing C by $C+\lambda$ yields the same results.) Fractional powers of these operators have been studied by various authors, a particularly useful reference being Chapter IV of SZ.-NAGY and FOIAŞ [21] (cf. also SZ.-NAGY and FOIAŞ [20] and [22]). In [17] LIONS has proven (cf. also KATO [13], KATO [14] and FOIAŞ and LIONS [7]) that for $0\leqq\tau\leqq 1$, $D(C^\tau)=D(|C|^\tau)$ and likewise $D(C^{*\tau})=D(|C^*|^\tau)$. It is known, [12] and [21], Theorem 5.1, that $D(C^\tau)=D(C^{*\tau})$ for $0\leqq\tau<\frac{1}{2}$. In Théorème 6.1 of [17], LIONS has given conditions implying that $D(C^{1/2})=D(C^{*1/2})$, and then shown that these conditions are satisfied for a large class of elliptic boundary value problems under sufficient regularity conditions.

In this section another sufficient condition for the equality $D(C^{1/2})=D(C^{*1/2})$ will be proven. It will then be shown that the condition is satisfied in the case of the Dirichlet problem with homogeneous boundary data on Lipschitzian graph domains (cf. [2], § 11).

Theorem 2. *If there exists a Hilbert space W such that*

i) $W\subsetneq_c D(C)$, $W\subsetneq_c D(C^*)$, *and* ii) $V_a\subset[W, H]_{1/2}$,

then $D(C^{1/2}) = D(C^{*1/2}) = V_a$.

Proof. By i) the identity mapping is continuous from W into $D(C)$, continuous from W into $D(C^*)$, and continuous from H into H. Therefore the quadratic interpolation theorem of [16], pp. 431—432, yields $[W, H]_{1/2}\subsetneq_c D(|C|^{1/2})$ and $[W, H]_{1/2}\subsetneq_c D(|C^*|^{1/2})$. Thus ii) and the preceding remarks yield $V_a\subset D(C^{1/2})$ and $V_a\subset D(C^{*1/2})$. The theorem now follows from Corollaire 5.1 of [17] or the Corollary of page 243, [14].

Now let $D\subset R^n$ be a Lipschitzian graph domain and let m be a positive integer. Denote the closure of $C_0^\infty(D)$ in $\check{P}^m(D)$ by $\check{P}_0^m(D)$. For $u, v\in\check{P}_0^m(D)$ let

$$c(u, v) = \sum_{|i|, |j|\leqq m}\int_D c_{ij}(x)D_j u\,\overline{D_i v}\,dx$$

with $c_{ij}\in C^{|i|}(\bar{D})$ where $C^{|i|}(\bar{D})$ here means the class of functions with all partial derivatives of order $\leqq|i|$ continuous and bounded on $\bar{D}$. Further assume that there is a $\gamma>0$ such that

$$\mathrm{Re}\, c(v, v)\geqq\gamma|v|^2_{m, D}\quad\text{for all}\quad v\in\check{P}_0^m(D).$$

Now let $H=L^2(D)$, $V_a=\check{P}_0^m(D)$, and $W=\check{P}_0^{2m}(D)$. It is easily verified (as e.g. in GREENLEE [8], § 6) that $W\subsetneq_c D(C)$ and $W\subsetneq_c D(C^*)$. Moreover, by Theorem 5.2 of [8], $[\check{P}_0^{2m}(D), L^2(D)]_{1/2}=[W, H]_{1/2}$ and $\check{P}_0^m(D)=V_a$ coincide with equivalent norms. Thus by Theorem 2, $D(C^{1/2})=D(C^{*1/2})=\check{P}_0^m(D)$.

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(Received May 12, 1972)