Bernstein-type inequalities for families of multiplier operators in Banach spaces with Cesàro decompositions. II. Applications*)

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In this paper a number of applications of the results in Part I is given by studying certain concrete instances of Banach spaces X and systems $\{P_k\}$ of orthogonal projections. Rather than to give a complete list of possible applications, our aim is to show how the general approach proposed in Part I yields Bernstein-type inequalities for classical orthonormal systems such as those concerned with Bessel, Laguerre, Hermite and ultraspherical polynomials, Walsh and Haar functions, and spherical harmonics. Let us mention that the present unifying approach covers certain classical as well as a number of new Bernstein-type inequalities.

In the following, $L_w^p(a, b)$, $1 \leqq p \leqq \infty$, $-\infty \leqq a < b \leqq +\infty$, denotes the usual Banach space of measurable functions, pth power integrable with respect to the weight $w(x) \geqq 0$:

$$\|f\|_{p,w} = \Big\{\int_a^b |f(x)|^p w(x)\,dx\Big\}^{1/p}, \quad \|f\|_{\infty,w} = \operatorname*{ess\,sup}_{a \leqq x \leqq b} |f(x)|\,w(x);$$

in the case $w(x)=1$ we abbreviate to L^p, $\|f\|_p$.

4. Bessel series

Let $a=0$, $b=1$, and $w(x)=1$. Denoting by $J_\nu(x)$ the Bessel function of the first kind of order $\nu > -1$ and by $\{c_k\}_{k \in \mathbf{N}}$ the sequence of positive zeros of $J_\nu(x)$, arranged in ascending order of magnitude, the functions

$$\varphi_k^{(\nu)}(x) = (2x)^{1/2}(J_{\nu+1}(c_k))^{-1} J_\nu(c_k x) \quad (k \in \mathbf{N})$$

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form an orthonormal system on (0, 1). Thus the projections $P_k^{(\nu)}$, defined by

$$(P_k^{(\nu)}f)(x) = \Big[\int_0^1 f(u)\varphi_k^{(\nu)}(u)\,du\Big]\,\varphi_k^{(\nu)}(x) \quad (k\in\mathbf{N})$$

are mutually orthogonal. Wing [42] has shown that $\{\varphi_k^{(\nu)}\}$ forms a Schauder basis in $L^p(0, 1)$, $1<p<\infty$, for $\nu\geqq -1/2$, and BENEDEK and PANZONE [18] have extended this result to $-1<\nu<-1/2$ provided $1/(\nu+3/2)<p<1/(-\nu-1/2)$; moreover, these bounds are sharp.

By (3.6) one then has

Corollary 4.1. *Let $f\in L^p(0, 1)$ with ν, p specified as above. Then*

$$\Big\|\sum_{k=1}^{n} k^{\omega} P_k^{(\nu)} f\Big\|_p \leqq An^{\omega}\Big\|\sum_{k=1}^{n} P_k^{(\nu)} f\Big\|_p \quad (\omega>0), \tag{4.1}$$

the constant A being independent of $n\in\mathbf{N}$ and f.

In case $\nu=\pm 1/2$, this inequality reduces to the standard Bernstein inequality for trigonometric polynomials (cf. (3.15)) since $J_{1/2}(x)=[2/(\pi x)]^{1/2}\sin x$ and $J_{-1/2}(x)=[2/(\pi x)]^{1/2}\cos x$. Clearly, inequalities corresponding to (3.5), (3.7), (3.9)—(3.11) may also be formulated.

To give a classical interpretation of $B^{\{k^\omega\}}$ let us consider the differential operator $D_{(\nu)}$ defined by

$$(D_{(\nu)}f)(x) = f''(x)-[(\nu^2-1/4)/x^2]f(x).$$

Then the Liouville normal form of the Bessel differential equation

$$(xu'(x))'+(\lambda x-\nu^2/x)u(x) = 0 \quad (0<x\leqq 1) \tag{4.2}$$

reads $D_{(\nu)}f+\lambda f=0$, and $\varphi_k^{(\nu)}$ is just the eigenfunction of $D_{(\nu)}$ corresponding to the eigenvalue $\lambda=-k^2$, $k\in\mathbf{N}$. Then (4.1) gives

$$\|D_{(\nu)}f\|_p \leqq An^2\|f\|_p \quad (n\in\mathbf{N}) \tag{4.3}$$

for all

$$f\in \bigoplus_{k=1}^{n} P_k^{(\nu)}(L^p(0, 1)).$$

Analogously, one may consider the system of eigenfunctions $\psi_k^{(\nu)}$ of (4.2), namely

$$\psi_k^{(\nu)}(x) = \sqrt{2}\,(J_{\nu+1}(c_k))^{-1}J_\nu(c_k x) \quad (k\in\mathbf{N}).$$

They form an orthonormal system on (0, 1) with respect to the weight $w(x)=x$. Thus the projections $\tilde{P}_k^{(\nu)}$, defined by

$$(\tilde{P}_k^{(\nu)}f)(x) = \Big[\int_0^1 f(u)\psi_k^{(\nu)}(u)\,u\,du\Big]\,\psi_k^{(\nu)}(x) \quad (k\in\mathbf{N}),$$

are mutually orthogonal. Now $\{\psi_k^{(\nu)}\}$ is a Schauder basis in $L_w^p(0, 1)$ for $4/3<p<4$ in case $\nu \geqq -1/2$ (see [42]) and for $2/(2+\nu)<p<-2/\nu$ in case $-1<\nu<-1/2$ (see [18]); again the bounds are sharp. Thus, letting

$$(\tilde{D}_{(\nu)}f)(x) = (xf'(x))' - (\nu^2/x)f(x),$$

(3.6) delivers

$$\|\tilde{D}_{(\nu)}f\|_{p,w} \leqq An^2 \|f\|_{p,w} \quad (n \in \mathbf{N}) \tag{4.4}$$

for each $f \in \bigoplus_{k=1}^{n} \tilde{P}_k^{(\nu)}(L_w^p(0, 1))$. Clearly (4.3) and (4.4) are equivalent in case $p=2$.

Similarly, using results of BENEDEK and PANZONE [17] and GENEROZOV [30], Bernstein-type inequalities corresponding to the eigenfunctions of the equation $(x^{2\varkappa}v'(x))' + \lambda v(x) = 0$ with $-\infty<\varkappa<1$ may be obtained. Moreover, results of RUTOVITZ and CRUM cited in [17] allow one to apply the present method to the eigenfunctions of a certain general class of Sturm—Liouville problems.

5. Laguerre and Hermite series

Let $a=0$, $b=\infty$, and $w(x)=1$. Consider the Laguerre polynomials $L_k^{(\alpha)}$ of order $\alpha>-1$ defined by

$$L_k^{(\alpha)}(x) = (k!)^{-1} e^x x^{-\alpha} (d/dx)^k (e^{-x} x^{k+\alpha}) \quad (k \in \mathbf{P}).$$

Setting

$$\varphi_k^{(\alpha)}(x) = \left\{ \Gamma(\alpha+1) \binom{k+\alpha}{k} \right\}^{-1} x^{\alpha/2} e^{-x/2} L_k^{(\alpha)}(x),$$

the projections

$$(P_k^{(\alpha)}f)(x) = \left[\int_0^\infty f(u) \varphi_k^{(\alpha)}(u)\, du \right] \varphi_k^{(\alpha)}(x)$$

are mutually orthogonal. The system $\{P_k^{(\alpha)}\}_{k\in\mathbf{P}}$ satisfies (2.7) for $j=0$ in case $4/3<p<4$, $\alpha>-1$ (see ASKEY—WAINGER [15], MUCKENHOUPT [34]), and for $j=1$ in case $1 \leqq p \leqq \infty$, $\alpha>0$ or $(1+\alpha/2)^{-1}<p<-2/\alpha$, $-1<\alpha\leqq 0$ (see POIANI [36]). Hence by (3.7)

Corollary 5.1. *Let $f \in L^p(0, \infty)$ with α, p specified as above for $j=1$, and $\omega>0$. Then*

$$\left\| \sum_{k=0}^{n} \log(1+k^\omega) P_k^{(\alpha)} f \right\|_p \leqq A \log(1+n^\omega) \left\| \sum_{k=0}^{n} P_k^{(\alpha)} f \right\|_p \quad (n \in \mathbf{P}).$$

Since the $\varphi_k^{(\alpha)}$ are eigenfunctions of the differential operator

$$D_{(\alpha)} = \frac{d}{dx}\left(x \frac{d}{dx}\right) + \frac{\alpha+1}{2} - \frac{x}{4} - \frac{\alpha^2}{4x}$$

with eigenvalues $-k$, $k \in \mathbf{P}$, one has by (3.6) that for all $f \in L^p(0, \infty)$, p, α as specified above ($j=1$),

$$\left\| D_{(\alpha)}\left(\sum_{k=0}^{n} P_k^{(\alpha)} f\right)\right\|_p \leqq An \left\|\sum_{k=0}^{n} P_k^{(\alpha)} f\right\|_p \quad (n \in \mathbf{P}).$$

A consideration of the $L_k^{(\alpha)}(x)$ themselves in the space $L_w^p(0, \infty)$ with weight $w(x)=x^\alpha e^{-x}$ does not apply here since the $L_k^{(\alpha)}$ do not yield a (C,j)-basis for any $j \in \mathbf{P}$ except for the case $p=2$ (see POLLARD [37], ASKEY—HIRSCHMAN [14]).

Now let $a=-\infty$, $b=+\infty$, and $w(x)=1$. Consider the Hermite polynomials

$$H_k(x) = (-1)^k e^{x^2}(d/dx)^k e^{-x^2} \quad (k \in \mathbf{P}).$$

Setting

$$\varphi_k(x) = \left(2^k k! \sqrt{\pi}\right)^{-1/2} e^{-x^2/2} H_k(x),$$

$\{\varphi_k\}$ is an orthonormal family of functions on $(-\infty, \infty)$. Thus the projections

$$(P_k f)(x) = \left\{\int_{-\infty}^{\infty} f(u)\varphi_k(u)\,du\right\}\varphi_k(x)$$

are mutually orthogonal. The system $\{P_k\}_{k\in\mathbf{P}}$ satisfies (2.7) for $j=1$ in case $1 \leqq p \leqq \infty$ (see [28a], [36]). Since the φ_k are eigenfunctions of the differential operator $(d^2/dx^2)+$ $+(1-x^2)$ with eigenvalues $-2k$, $k \in \mathbf{P}$, one has by (3.6)

$$\|(d^2/dx^2)f + (1-x^2)f\|_p \leqq An\|f\|_p$$

for all $f \in \bigoplus_{k=0}^{n} P_k(L^p(-\infty, \infty))$, $1 \leqq p \leqq \infty$, $n \in \mathbf{P}$. This inequality is contained in a paper of FREUD [28].

6. Ultraspherical series

Let $a=-1$, $b=1$, and $w(x)=1$. The ultraspherical polynomials C_k^λ of order $\lambda \geqq 0$ are given by

$$C_k^\lambda(x) = M_{k,\lambda}(1-x^2)^{-\lambda+1/2}(d/dx)^k[(1-x^2)^{k+\lambda-1/2}] \quad (k \in \mathbf{P}), \tag{6.1}$$

$M_{k,\lambda}$ being a suitable constant. They are orthonormal on $(-1, 1)$ with respect to the measure $(1-x^2)^{\lambda-1/2}\,dx$. Hence, setting

$$\varphi_k^\lambda(x) = (1-x^2)^{\lambda/2-1/4} C_k^\lambda(x),$$

$$(P_k^{(\lambda)} f)(x) = \left[\int_{-1}^{1} f(u)\varphi_k^\lambda(u)\,du\right]\varphi_k^\lambda(x),$$

the projections $P_k^{(\lambda)}$ are mutually orthogonal on $L^p(-1, 1)$. The sequence $\{\varphi_k^\lambda\}_{k\in\mathbf{P}}$ forms a Schauder basis in $L^p(-1, 1)$ for $4/3<p<4$, $\lambda \geqq 0$ (cf. WING [42] for Jacobi polynomials). The functions φ_k^λ are eigenfunctions of the operator

$$D_{(\lambda)} = \frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right] + \left(\lambda - \frac{1}{2}\right)\left[1 - \left(\lambda - \frac{1}{2}\right)\frac{x^2}{1-x^2}\right]$$

with corresponding eigenvalues $-k(k+2\lambda)$, $k \in \mathbf{P}$. Thus by (3.5)

Corollary 6.1. *For any* $f \in L^p(-1, 1)$, $4/3 < p < 4$,

$$\left\| \sum_{k=0}^{n} e^{k(k+2\lambda)} P_k^{(\lambda)} f \right\|_p \leqq A e^{n(n+2\lambda)} \left\| \sum_{k=0}^{n} P_k^{(\lambda)} f \right\|_p . \tag{6.2}$$

Noting that the C_k^λ are orthonormal on $(-1, 1)$ with respect to $w(x) = (1-x^2)^{\lambda-1/2}$, set

$$(\tilde{P}_k^{(\lambda)} f)(x) = \Big[\int_{-1}^{1} f(u) C_k^\lambda(u) (1-u^2)^{\lambda-1/2} \, du \Big] C_k^\lambda(x) \quad (k \in \mathbf{P}).$$

The $\tilde{P}_k^{(\lambda)}$ form a (C, j)-decomposition in $L_w^p(-1, 1)$ provided

$$\begin{cases} \dfrac{2\lambda+1}{\lambda+1+j} < p < \dfrac{2\lambda+1}{\lambda-j} & \text{if} \quad 0 \leqq j \leqq \lambda,\ 0 \leqq \lambda < \infty \\ 1 \leqq p \leqq \infty & \text{if} \quad 0 \leqq \lambda < j \end{cases} \tag{6.3}$$

(see Pollard [37] for $j=0$, Askey—Hirschman [14] for $j>0$). The C_k^λ are eigenfunctions of

$$\tilde{D}_{(\lambda)} = (1-x^2)(d^2/dx^2) - (2\lambda+1) x (d/dx)$$

with eigenvalues $-k(k+2\lambda)$ so that by (3.6)

Corollary 6.2. *Let* $\mathscr{P}_n$ *denote the set of all algebraic polynomials of degree* $\leqq n$. *Then*

$$\|\tilde{D}_{(\lambda)} f\|_{p,w} \leqq A n^2 \|f\|_{p,w} \quad (f \in \mathscr{P}_n,\ n \in \mathbf{P}), \tag{6.4}$$

provided (6.3) *is satisfied with* $j=1$.

So far we have stated Bernstein inequalities of type (3.1), (3.3). However, those of Corollary 3.3 are valid as well. For example, by (3.11)

Corollary 6.3. *The Riesz means* (3.8) (iii) *of order* $\varkappa$, $\nu > 0$ *satisfy*

$$\left\| \sum_{k=0}^{n} k^\omega \left(1 - \left(\frac{k}{n+1}\right)^\varkappa\right)^\nu \tilde{P}_k^{(\lambda)} f \right\|_{p,w} \leqq A n^\omega \|f\|_{p,w} \quad (n \in \mathbf{P})$$

for arbitrary $\omega > 0$, $f \in L_w^p(-1, 1)$, *provided* (6.3) *holds for some* $0 \leqq j \leqq \nu$.

Remark. It is possible to extend the above results to Jacobi polynomials. Indeed, (6.2) may immediately be restated since [42] includes $(C, 0)$-summability for the Jacobi case for $4/3 < p < 4$. Also (C, 0)-summability for Jacobi series in the weight space $L_w^p(-1, 1)$ with $w(x) = (1-x)^\alpha (1+x)^\beta$, $\alpha, \beta > -1$, is known (see Pollard [38] and Muckenhoupt [33]) so that the Jacobi analogue of (6.4) follows, namely

$$\|\tilde{D}_{(\alpha,\beta)} f\|_{p,w} \leqq A n (n+\alpha+\beta+1) \|f\|_{p,w} \quad (f \in \mathscr{P}_n,\ n \in \mathbf{P}), \tag{6.5}$$

where $\tilde{D}_{(\alpha,\beta)}$ is defined by

$$\tilde{D}_{(\alpha,\beta)} = (1-x^2)(d^2/dx^2) + \big(\beta - \alpha - (\alpha+\beta+2) x\big)(d/dx)$$

and p is restricted to

$$(\alpha+1)\left(\frac{1}{p}-\frac{1}{2}\right)<\min\left\{\frac{1}{4}, \frac{1+\alpha}{2}\right\}, \ (\beta+1)\left|\frac{1}{p}-\frac{1}{2}\right|<\min\left\{\frac{1}{4}, \frac{1+\beta}{2}\right\}$$

A more general inequality, which contains (6.5) and the $w(x)=1$-analogue of (6.5) as particular instances, can be stated as well using the result of [42] and [33].

Concerning (C, j)-summability in $L_w^p(-1, 1)$ one may proceed via POLLARD [38], STEIN [41], ASKEY—HIRSCHMAN [14], ASKEY—WAINGER [16], and GASPER [29], according to a written communication of R. Askey.

The paper of STEIN [41] should also be mentioned in connection with (6.5) since it contains a proof for all $1 \leqq p \leqq \infty$, $\alpha, \beta > -1$. He also assumes condition (2.7) to be valid for some $j \in \mathbf{P}$ and obtains Bernstein-type inequalities for orthonormal systems, using a different method, namely interpolation in polynomial subspaces.

7. Walsh series

Let $a=0$, $b=1$, and $w(x)=1$, all functions in this section being assumed to have period 1. Defining the Rademacher functions by

$$\varphi_0(x) = \begin{cases} 1, & 0 \leqq x < 1/2, \\ -1, & 1/2 \leqq x < 1, \end{cases} \quad \varphi_0(x+1) = \varphi_0(x),$$

$$\varphi_k(x) = \varphi_0(2^k x) \quad (k \in \mathbf{N}),$$

the Walsh functions are given by

$$\psi_0(x) = 1, \quad \psi_k(x) = \varphi_{k_1}(x)\varphi_{k_2}(x) \dots \varphi_{k_i}(x),$$

$$k = 2^{k_1} + \dots + 2^{k_i}, \quad k_1 > \dots > k_i \geqq 0, \quad k_i \in \mathbf{P}.$$

They form an orthonormal system in $L^p(0, 1)$, $1 \leqq p < \infty$, which is also fundamental. Thus the projections

$$(P_k f)(x) = \left[\int_0^1 f(u)\psi_k(u)\,du\right]\psi_k(x) \quad (k \in \mathbf{P})$$

are mutually orthogonal and total in $L^p(0, 1)$, $1 \leqq p < \infty$; PALEY [35] has shown that the P_k form a Schauder decomposition of $L^p(0, 1)$ for $1 < p < \infty$; for the proof that they also form a $(C, 1)$-decomposition of $L^1(0, 1)$ see e.g. FINE [27], MORGENTHALER [32]. Hence, by (3.9)

$$\text{(7.1)} \qquad \left\| \sum_{k=0}^{\infty} k^{\omega} e^{-(k/\varrho)^{\varkappa}} P_k f \right\|_p \leqq A \varrho^{\omega} \|f\|_p \quad (\varkappa, \omega > 0)$$

for any $f \in L^p(0, 1)$, $1 \leqq p < \infty$, $\varrho > 0$.

As in the preceding sections we would like to interpret the case $\omega=1$ via some differential operator. Concerning its definition we follow up BUTZER—WAGNER [21, 22]: Let G denote the dyadic group consisting of all sequences $x=\{x_n\}_{n=1}^{\infty}$ such that $x_n=0$ or 1, the operation of G being termwise addition mod 2. A function $f\in L^p(0, 1)$ is said to have a strong derivative $D_G f$ in $L^p(0, 1)$ if there exists $g\in L^p(0, 1)$ such that

$$\lim_{m\to\infty}\left\|\frac{1}{2}\sum_{i=0}^{m}2^i[f(\cdot)-f(\cdot\otimes 2^{-i-1})]-g(\cdot)\right\|_p = 0,$$

and in this case $D_G f=g$. Here

$$x\otimes y = \frac{1}{2}\sum_{n=1}^{\infty}|x_n-y_n|, \quad x=\sum_{n=1}^{\infty}x_n 2^{-n}, \quad y=\sum_{n=1}^{\infty}y_n 2^{-n},$$

finite expansions being used for dyadic rationals. The operator D_G is closed and linear (see [21, Prop. 4.4] and [22, Sec. 3], where further details are given). Since the Walsh functions satisfy $D_G\psi_k=k\psi_k$ for each $k\in\mathbf{P}$, i.e. the ψ_k are eigenfunctions of D_G, one has by (7.1), (3.8) (i)

$$\|D_G W_\varkappa(\varrho)f\|_p \leqq A\varrho\,\|f\|_p$$

for all $f\in L^p(0, 1)$, $1\leqq p<\infty$, of period 1.

8. Haar series

Let $a=0$, $b=1$, and $w(x)=1$. In the notation of [8, p. 49] the orthonormal system $\{h_k(x)\}_{k=1}^{\infty}$ of Haar functions is defined on [0, 1] by

$$h_1(x) = \chi_{[0,1]}(x)$$
$$h_k(x) = 2^{m/2}\{\chi_{[0,1]}(2^{m+1}x-2k+2)-\chi_{(0,1]}(2^{m+1}x-2k+1)\},$$

where $k=2^m+i$, $m\in\mathbf{P}$, $i=1, 2, \ldots, 2^m$, and $\chi_{[a,b]}(x)$ denotes the characteristic function of the interval $[a, b]$. Hence the projections P_k defined by

$$(P_k f)(x) = \Big[\int_0^1 f(u)h_k(u)\,du\Big]\,h_k(x) \quad (k\in\mathbf{N})$$

are mutually orthogonal. Moreover, the Haar functions form a Schauder basis in $L^p(0, 1)$, $1\leqq p<\infty$ (see also [11, p. 13]) so that one has as an immediate consequence of Corollary 3.2

$$\text{(8.1)} \qquad \left\|\sum_{k=1}^{n}\alpha_k P_k f\right\|_p \leqq A\alpha_n\left\|\sum_{k=1}^{n}P_k f\right\|_p \quad (f\in L^p(0, 1)),$$

where α may be any of the examples (3.4).

An explicit definition of a genuine differential operator D satisfying $Dh_k=kh_k$ for all $k\in\mathbf{N}$ seems to be unknown. Nevertheless such an operator can be identified with the infinitesimal generator of a suitable semi-group of class $(\mathscr{C}_0)$. For example, let the Abel—Poisson means $W(t)$ of the Haar expansion of f be defined by

$$(W(t)f)(x) = \sum_{k=1}^{\infty} e^{-tk} P_k f \quad (t > 0);$$

then $\lim_{t\to 0+} \|W(t)f-f\|_p=0$ for $1\leqq p<\infty$ (cf. [2II]). Setting $W(0)=I$, it follows that $\{W(t),\, t\geqq 0\}$ is a semi-group of bounded linear operators on $L^p(0,1)$ of class $(\mathscr{C}_0)$. Its infinitesimal generator $\mathscr{A}$ is easily seen to be represented by

$$\mathscr{A}f \sim \sum_{k=1}^{\infty} (-k) P_k f$$

for every f in the domain $D(\mathscr{A})$ of $\mathscr{A}$. Moreover (cf. [20, p. 9]), $D(\mathscr{A})$ is dense in X, and $\mathscr{A}$ is a closed linear operator. Thus $-\mathscr{A}$ is just the desired operator D. Differential operators corresponding to the logarithmic and exponential cases in (8.1) may be defined similarly (see [7]).

Clearly in certain instances the semi-group theory yields directly Bernstein-type inequalities in an arbitrary Banach space X. Indeed, for holomorphic semi-groups of class $(\mathscr{C}_0)$ on X with infinitesimal generator $\mathscr{A}$ one always has the inequality $\|\mathscr{A}T(t)f\|_X\leqq Mt^{-1}\|f\|_X$ for all $f\in X$, $t>0$ by Cauchy's integral formula (see Butzer—Berens [20, Sec. 1.1.2]).

Along the present lines one may also treat generalized Schauder systems (cf. Canturija [23]), generalized Haar systems (cf. Golubov [31], Sox—Harrington [40]), the (orthonormalized) Franklin system (cf. Ciesielski [24, 25], Radecki [39]) as well as further spline function systems (cf. Ciesielski—Domsta [26]). The Bernstein-type inequalities obtained in [24, 39, 26] deal with ordinary derivatives which, however, are not covered by our approach.

9. Spherical harmonics

Let $\mathbf{R}^N$ be the N-dimensional Euclidean space ($N\geqq 2$) with elements $v=(v_1, \dots, v_N)$, inner product $v\cdot v^*=\sum_{k=1}^{N} v_k v_k^*$, and $|v|^2=v\cdot v$. Denoting by S_N the surface of the unit sphere in R^N with elements y, z, content $\Omega_N=2\pi^{N/2}/\Gamma(N/2)$ and surface element ds, let $X(S_N)$ be one of the spaces $L^p(S_N)$, $1\leqq p<\infty$, or $C(S_N)$ with norms

$$\|f\|_p = \Big\{\Omega_N^{-1} \int_{S_N} |f(y)|^p \, ds(y)\Big\}^{1/p} (1 \leqq p < \infty), \quad \|f\|_C = \max_{y\in S_N} |f(y)|,$$

respectively. If $Y_k(v)$ is a homogeneous polynomial of degree k satisfying

$$\Delta Y_k(v) = 0, \quad \Delta = \sum_{k=1}^{N} (\partial/\partial v_k)^2 \quad (v \in \mathbf{R}^N),$$

then the restriction of Y_k to S_N, denoted by Y_k, too, is called a surface spherical harmonic of degree k. The Y_k satisfy the differential equation

$$\tilde{\Delta} Y_k(y) = -k(k+N-2) Y_k(y), \quad \tilde{\Delta} f(v) = |v|^2 \Delta f(v/|v|). \tag{9.1}$$

Let the orthonormal sequence of projections $\{P_k\}_{k\in \mathbf{P}}$ be defined by (cf. (6.1), $\lambda=(N-2)/2$)

$$(P_k f)(y) = \sum_{r=1}^{H(k,N)} \Big\{ \int_{S_N} f(z) Y_k^r(z)\, ds(z) \Big\} Y_k^r(y) = \frac{\Gamma(\lambda)(k+\lambda)}{2\pi^{\lambda+1}} \int_{S_N} C_k^\lambda(y\cdot z) f(z)\, ds(z),$$

where $H(k, N)$ denotes the number of linearly independent spherical harmonics of degree k. The P_k form a (C, j)-decomposition of $X(S_N)$ for $j>(N-2)/2$ (see [19] and the literature cited there).

Since, for $\varkappa>2$, $\{k(k+N-2)n^{-2}[1+(k/n)^2]^{-\varkappa/2}\}_{k\in\mathbf{P}} \in bv_{j+1}$ uniformly in $n\in\mathbf{P}$, it follows by (3.10), (9.1) that

Corollary 9.1. *Let $f\in X(S_N)$. Then for any $\varkappa>2$*

$$\|\tilde{\Delta} L_\varkappa(x) f\|_X \leqq A n^2 \|f\|_X.$$

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