The convolution theorems of Dieudonné

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Let G denote an arbitrary locally compact abelian group that is not discrete. In [1], DIEUDONNÉ showed that if $f \in L^1(G)$ then $f * L^1(G)$ must be a proper subset of $L^1(G)$. His proof involves a rather complicated construction of some particular functions on G.

In this note we prove a general result (Theorem B) about Banach algebras with elements which are generalized divisors of zero, and deduce Dieudonné's result as an easy corollary (Theorem D). An extension to Banach modules yields his result that $f * L^q(G) \neq L^q(G)$ if $f \in L^1(G)$, $1 < q < \infty$.

Definition A. Let B be a commutative normed algebra. The element $f \in B$ is said to be a *generalized divisor of zero* (gdz) if there exists a sequence $\{g_n\}$ in B such that

$$\|g_n\| = 1 \quad (n = 1, 2, \ldots), \quad \text{but} \quad fg_n \to 0.$$

This is equivalent to the definition in [2, p. 69] which treats only algebras with unit.

Theorem B. *Let B be a commutative, semi-simple Banach algebra. If $f \in B$ is a* gdz *then $fB \neq B$.*

Proof. We may assume that the Gelfand transform $\hat{f}$ of f never vanishes. For, if $\hat{f}(\lambda)=0$ for some λ, then $\hat{g}(\lambda)=0$ for every $g \in fB$ so that $fB \neq B$

Define $T: B \to B$ by $Tg = fg \quad (g \in B)$. Then T is $1-1$. For if $Tg=fg=0$, then $\hat{f}\hat{g}=0$. Since $\hat{f}$ never vanishes this implies $\hat{g}=0$, and so $g=0$ since B is semi-simple. T is clearly continuous. We wish to show that T is not onto.

Assume the contrary. Then T is $1-1$, continuous, and onto. The inverse function theorem for Banach spaces then implies that T^{-1} is continuous. Since f is a gdz there exists $\{g_n\}$ with $fg_n \to 0$ and

$$(1) \qquad \|g_n\| = 1 \quad (n = 1, 2, \ldots).$$

Since T^{-1} is continuous we have $T^{-1}(fg_n) \to 0$. But $fg_n = Tg_n$ so that $T^{-1}(fg_n) = g_n$. Hence $g_n \to 0$ which contradicts (1). The contradiction shows that T is not onto which is what we wished to show.

We next show that every element f in $L^1(R)$ is a gdz. Let

$$g_n(t) = e^{int}\delta(t) \quad (-\infty < t < \infty; \quad n = 1, 2, \dots)$$

where δ is any bounded L^1 function with $\|\delta\|_1 = 1$. Then $\|g_n\|_1 = 1$. Now for each t the function $f(u)\delta(t-u)$ is in L^1. Also

$$f*g_n(t) = e^{int} \int_{-\infty}^{\infty} e^{-inu} f(u)\delta(t-u)\,du$$

which tends to zero for each t as $n \to \infty$ by the Riemann—Lebesgue theorem. Moreover, $f*g_n$ is dominated by

$$\int_{-\infty}^{\infty} |f(u)\delta(t-u)|\,du$$

which is integrable since $f, \delta \in L^1$. Hence

$$\|f*g_n\|_1 \to 0$$

by the Lebesgue dominated convergence theorem. Thus $\|g_n\|_1 = 1$ and $f*g_n \to 0$ which shows that f is a gdz. (Note, too, that $\{g_n\}$ is independent of f.)

The above argument extends easily to any non-discrete G. Simply replace the functions e^{int} by characters $\chi_n(t)$ on G where $\{\chi_n\}$ tends to infinity on $\hat{G}$ (which is not compact). Thus we have

Theorem C. *If G is a locally compact abelian group that is not discrete, then every element of $L^1(G)$ is a* gdz.

Here is Dieudonné's result.

Theorem D. *If G is a locally compact abelian group that is not discrete, then $f*L^1(G) \neq L^1(G)$ for all $f \in L^1(G)$.*

Proof. The space $L^1(G)$ is a commutative semi-simple Banach algebra. If $f \in L^1(G)$ then, by Theorem C, f is a gdz. Hence, by Theorem B, $f*L^1(G) \neq L^1(G)$.

Dieudonné actually proved that $f*L^q \neq L^q$ for every $f \in L^1$, $1 < q < \infty$. To include this result we generalize to modules. See [3, p. 263] for the definition of a Banach A-module. The example that will interest us is L^q which is a Banach L^1-module.

Definition E. Let A be a commutative Banach algebra and let B be a Banach A-module. The element $f \in A$ is said to be a *generalized divisor of zero with respect to B* (abbreviate gdz-B) if there exists a sequence $\{g_n\}$ in B such that

$$\|g_n\|_B = 1 \quad (n = 1, 2, \dots), \quad \text{but} \quad \|fg_n\|_B \to 0.$$

Completeness is not essential in the above definition but it is in what follows.

Because B in the following theorem need not be an algebra, we do not have the Gelfand transform available. We introduce a $1-1$ hypothesis that was not necessary in Theorem B, and we will handle the non $1-1$ case specially when we come to L^q.

Theorem F. *Let A be a commutative Banach algebra and let B be a Banach A-module. If $f \in A$ is a* gdz-*B, and if T: $B \to B$ defined by $Tg = fg$ $(g \in B)$ is $1-1$, then T is not onto. That is, $fB \neq B$.*

Proof. Same as that of Theorem B.

Corresponding to Theorem C we have

Theorem G. *If G is as in Theorem* C *then every $f \in L^1(G)$ is a* gdz-$L^q(G)$.

Proof. Again, the case $G=R$ tells all. Take any $f \in L^1(R)$. Let

$$g_n(t) = e^{int}\delta(t) \quad (-\infty < t < \infty; \quad n = 1, 2, \ldots)$$

where δ is a bounded L^q function with $\|\delta\|_q = 1$. Then $f_n * g(t) \to 0$ for all t, as before. Moreover, $|f * g_n|^q$ is dominated by $(|f| * |\delta|)^q$. But $|f| * |\delta| \in L^q$ since $f \in L^1$, $\delta \in L^q$, so that $(|f| * |\delta|)^q$ is integrable. Thus $\|f * g_n\|_q \to 0$ by the dominated convergence theorem. Since $\|g_n\|_q = 1$, the proof is complete.

Finally,

Theorem H. *If G is as in Theorem* C, *then*

$$f * L^q(G) \neq L^q(G) \quad \text{for all} \quad f \in L^1(G).$$

Proof. For $f \in L^1(G)$ let $E \subset \hat{G}$ be the set where $\hat{f} = 0$. We consider two cases.

a) If $mE = 0$ then the map $T: g \to f * g$ $(g \in L^q)$ is $1-1$. For if $Tg = f * g = 0$ then $\hat{f}\hat{g} = 0$ almost everywhere on $\hat{G}$. Since $mE = 0$ this implies $\hat{g} = 0$ a.e. and hence $g = 0$. The desired conclusion then follows from Theorems F and G.

b) If $mE > 0$ then, since the transform of every function in $f * L^q(G)$ vanishes a.e. on E, it is clear that $f * L^q(G) \neq L^q(G)$.

This argument is valid for $1 < q \leqq 2$. For $q > 2$ an easy adjoint argument applies.

References

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