Operators satisfying a sequential growth condition

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§ 1. Introduction

An operator T on a Hilbert space $\mathfrak{H}$ is called hyponormal if $T^*T-TT^* \geqq 0$. One of the very useful properties of a hyponormal operator T is that it satisfies the G_1 *growth condition*, that is $\|(\lambda-T)^{-1}\| = 1/d(\lambda)$ for all $\lambda \in \varrho(T)$ where $\sigma(T)$ and $\varrho(T)$ denote the spectrum and the resolvent set of T respectively, and $d(\lambda) = \text{dist}\,[\lambda, \sigma(T)]$. For most applications we need this growth condition to be satisfied in a neighborhood of $\sigma(T)$. On the other hand, the Volterra operator V does not satisfy the growth condition G_1 in any neighborhood of $\sigma(V)$, but there does exist a sequence $\lambda_n \in \varrho(V)$ (take λ_n to be negative real numbers) such that $\lambda_n \to 0$ and $\|(V-\lambda_n)^{-1}\| = 1/|\lambda_n|$. This motivates us to introduce the concept of a *sequential G_1 growth condition*. A bounded operator T on a Hilbert space $\mathfrak{H}$ satisfies sequential G_1 growth condition if for every $\lambda \in \partial(\sigma(T))$ (the boundary of $\sigma(T)$), there exists a sequence $\lambda_n \in \varrho(T)$ such that $\lambda_n \to \lambda$ and $\|(\lambda_n - T)^{-1}\| = 1/d(\lambda_n)$ for all n. Such an operator T is also referred to as a *sequentially G_1 operator*. Some other generalizations of G_1 growth conditions have been considered by LUCKE [5, 6] and RIGGS [8].

We prove that a sequentially G_1 algebraic operator is normal. This result has an interesting application to the theory of ϱ-dilations in the sense that it generalizes and at the same time simplifies the proof of a recent theorem of FURUTA [2] concerning C_ϱ-operators. We also prove that if T is a sequentially G_1 operator then $T+\mathcal{K} \subset \overline{\mathcal{R}}_1$ where $\mathcal{K}$ is the ideal of compact operators and $\overline{\mathcal{R}}_1$ denotes the norm closure of operators with one dimensional reducing subspace. Our result generalizes a theorem of BERBERIAN [1] and ISTRĂŢESCU [4] which asserts that $T+\mathcal{K} \subset \overline{\mathcal{R}}_1$ whenever T is a G_1 operator (this in turn is a generalization of a result of STAMPFLI [12] about hyponormal operators). $\mathcal{B}(\mathfrak{H})$ denotes the algebra of bounded linear operator on $\mathfrak{H}$.

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The following proposition shows the existence of a class of sequentially G_1 operators which are not G_1 operators.

Proposition 1. 1. *If $T\in\mathscr{B}(\mathfrak{H})$ is a quasi-nilpotent operator such that $0\in\partial w(T)$, where $w(T)=\{(Tx, x):x\in H$ and $\|x\|=1\}$ is the numerical range of T, then T is a sequentially G_1 operator.*

Proof. Since $w(T)$ is convex there is a line of support for $w(T)$ passing through 0 (since $0\in\partial w(T)$) and hence without loss of generality we can assume that $w(T)\subset \subset\{\lambda:\text{Real }\lambda\geqq 0\}$. It is quite easy to show that, for any $T\in\mathscr{B}(\mathfrak{H})$ and $\lambda\notin\overline{w(T)}$, $\|(\lambda-T)^{-1}\|\leqq\frac{1}{\operatorname{dist}[\lambda,\overline{w(T)}]}$. Since $0\in\partial w(T)$ and $\sigma(T)=\{0\}$, for any real negative number λ, $\operatorname{dist}[\lambda,\overline{w(T)}]=|\lambda|=d(\lambda)$. Hence we can take $\lambda_n=-1/n$ and then $\|(\lambda_n-T)^{-1}\|=\frac{1}{|\lambda_n|}$ for all n.

In view of a theorem of Stampfli [9], if T is a G_1 operator and if $\sigma(T)$ is a finite set, then T is a normal operator. Thus no non-zero quasi-nilpotent operator is a G_1 operator. Out next result shows that no non-zero nilpotent operator is a sequentially G_1 operator.

Proposition 1. 2. *Let $T\in\mathscr{B}(\mathfrak{H})$ be such that $T^m=0$ for some $m>1$ and suppose that T is a sequentially G_1 operator. Then $T=0$.*

Proof. Since T is a sequentially G_1 operator and $\sigma(T)=\{0\}$, there exists a sequence $\lambda_n\to 0$ such that

$\|(\lambda_n-T)^{-1}\|=\frac{1}{|\lambda_n|}$ for all n. Suppose $m>1$, then $(\lambda_n-T)^{-1}=\sum_{i=0}^{m-1}\frac{T^i}{\lambda_n^{i+1}}$ this implies $\frac{\|T^{m-1}\|}{|\lambda_n|^m}-\sum_{i=0}^{m-2}\frac{\|T^i\|}{|\lambda_n|^{i+1}}\leqq\frac{1}{|\lambda_n|}$ for all n. Hence $\|T^{m-1}\|\leqq|\lambda_n|^{m-1}+\sum_{i=0}^{m-2}\|T^i\|\,|\lambda_n|^{m-i-1}$, for all n. Since $|\lambda_n|^{m-1}+\sum_{i=0}^{m-2}\|T^i\|\,|\lambda_n|^{m-i-1}\to 0$ as $n\to\infty$, we conclude that $T^{m-1}=0$. Hence by a simple induction argument $T=0$. We thank the Referee for pointing out that this result holds even if $\|(\lambda_n-T)^{-1}\|\leqq\frac{M}{|\lambda_n|}$, $M\geqq 1$.

Corollary 1. 3. *Let $T\neq 0$ be a nilpotent operator, then $0\in$ Interior $w(T)$.*

§ 2. Sequentially G_1 operators and the class C_ϱ

An operator T is called *algebraic* if there exists a polynomial $p(z)$ such that $p(T)=0$. We assume that this $p(z)$ is minimal among all the polynomials $q(z)$ such that $q(T)=0$. We shall show that if T is a sequentially G_1 algebraic operator then T

is normal. To prove this result we need the following lemma, which appears implicitly in STAMPFLI [10] and explicitly in PUTNAM [7] and STAMPFLI [12].

Lemma 2. 1. (Putnam—Stampfli) *Let* $T\in\mathcal{B}(\mathfrak{H})$ *and let* $\lambda_0\in\sigma(T)$ *such that* $Tx=\lambda_0 x$, $\|x\|=1$. *Suppose there exists a sequence* $\{\lambda_n\}\in\varrho(T)$ *such that* $\lambda_n\to\lambda_0$ *and* $\lim_{n\to\infty}|\lambda_n-\lambda_0|\,\|(T-\lambda_n)^{-1}\|=1$; *then* $T^*x=\bar{\lambda}_0 x$.

Theorem 2. 2. *Let* $T\in\mathcal{B}(\mathfrak{H})$ *be a sequentially* G_1 *algebraic operator. Then* T *is normal.*

Proof. Since T is algebraic, there exists a polynomial $p(z)$ such that $p(T)=0$. Let z_i $(i=0,\ldots,m)$ be the distinct roots of $p(z)$ of multiplicity n_i $(i=0,\ldots,m)$ respectively. Then $\mathfrak{H}=\sum_{i=1}^{m}\eta_i$ where $\eta_i=\{x\in\mathfrak{H}:(T-z_i)^{n_i}x=0\}$. Thus each η_i is invariant under T and $\sigma(T|\eta_i)=\{z_i\}$. Since T is sequentially G_1, it follows that $T|\eta_i$ is sequentially G_1. Moreover $T-z_i|\eta_i$ is a nilpotent operator. Hence by Proposition 1. 2, $T-z_i|\eta_i=0$. Thus $\eta_i=\eta(T-z_i)\doteq$ null space of $T-z_i$. Moreover, by Lemma 2. 1, $\eta(T-z_i)=\eta(T^*-\bar{z}_i)$ and $\eta(T-z_i)\perp\eta(T-z_j)$ for $i\neq j$. Hence $T=\sum_{i=0}^{m}\oplus z_iP_i$ where P_i denotes the projection of $\mathfrak{H}$ onto $\eta(T-z_i)$ and T is normal.

The next theorem shows that the above hypothesis can be slightly changed without affecting the conclusion. The hypothesis in the following theorem means roughly that T is sequentially G_1 except at one point.

Theorem 2. 3. *Let* $T\in\mathcal{B}(\mathfrak{H})$ *such that* $p(T)=0$, *where* $p(z)=(z-z_0)(z-z_1)^{n_1}\ldots\ldots(z-z_m)^{n_m}$. *Suppose for each* z_i $(i=1,2,\ldots,m)$ *there exists a sequence* $\{\lambda_n^{(i)}\}_{n=0}^{\infty}\in\varrho(T)$ *such that* $\lambda_n^{(i)}\to z_i$ *and* $\|(\lambda^{(i)}-T)^{-1}\|=\dfrac{1}{|\lambda_n^{(i)}-z_i|}$ *for all* n. *Then* $\mathfrak{H}=\sum_{i=0}^{m}\oplus\eta(T-z_i)$ *and* T *is normal.*

Proof. From the proof of Theorem 2. 2, it follows that

$$\mathfrak{H}=\eta(T-z_0)\dotplus\sum_{i=1}^{m}\oplus\eta(T-z_i)\quad\text{and}\quad\eta(T-z_i)=\eta(T^*-\bar{z}_i)\quad\text{for}\quad i=1,2,\ldots,m.$$

Thus $\eta(T-z_0)$ is also orthogonal to $\eta(T-z_i)$ for $i=1,2,\ldots,m$. Hence T is normal.

Now we shall apply the above result to get a generalization of a result of FURUTA [2] about the operators in C_ϱ class. The *class* C_ϱ of operators was introduced by SZ.-NAGY and FOIAŞ [13] as the set of all operators T on a Hilbert space $\mathfrak{H}$ for which there exists a unitary operator U on some Hilbert space $\mathcal{K}$ $(\mathcal{K}\supset\mathfrak{H})$ such that

$$T^n=\varrho PU^n|\mathfrak{H}\qquad(n=1,2,\ldots),$$

where P is the projection of $\mathcal{K}$ onto $\mathfrak{H}$. U is called *unitary* ϱ*-dilation* of T.

One of the characterizations of the class C_ϱ, $\varrho \geqq 2$ is the following:

Theorem 2. 4. (SZ.-NAGY and FOIAŞ [14]) *An operator $T \in \mathscr{B}(\mathfrak{H})$ belongs to the class C_ϱ ($\varrho \geqq 2$) if and only if T satisfies the following condition*

$$\|(\mu - T)^{-1}\| \leqq \frac{1}{|\mu|-1} \quad \begin{array}{ll} \text{for} \quad 1<|\mu|<\infty & \text{if} \quad \varrho = 2 \\ \text{for} \quad 1<|\mu| \leqq \dfrac{\varrho-1}{\varrho-2} & \text{if} \quad \varrho>2. \end{array}$$

Theorem 2. 5. *Let $T \in C_\varrho$ ($\varrho>0$). Suppose $p(T)=0$ where $p(z)$ is a polynomial and all roots of $p(z)$ are on the unit circle except for, perhaps a simple root (say z_0). Then $T = U \oplus z_0 P$ where P is a projection of $\mathfrak{H}$ onto the null space of $T-z_0$, and U is a unitary operator.*

Proof. Since $C_\varrho \subset C_{\varrho'}$ for $0<\varrho<\varrho'$ ([14, page 50]), $T \in C_\varrho$ ($\varrho>0$) implies that $T \in C_{\varrho+2}$ and hence by Theorem 2. 4,

$$\|(\mu - T)^{-1}\| \leqq \frac{1}{|\mu|-1} \quad \text{for} \quad 1<|\mu|<\frac{\varrho+1}{\varrho}.$$

Let $p(z) = (z-z_0)(z-z_1)^{n_1} \ldots (z-z_m)^{n_m}$ where $|z_i|=1$ for $i=1, 2, \ldots, m$. Now for any μ, $1<|\mu|<1+\frac{1}{\varrho}$, μ collinear with z_i ($i=1, 2, \ldots, m$);

$$\|(\mu - T)^{-1}\| \leqq \frac{1}{|\mu|-|z_i|} = \frac{1}{|\mu - z_i|}$$

Hence $\|(\mu-T)^{-1}\| = \frac{1}{|\mu-z_i|} = \frac{1}{d(\mu)}$. Thus T satisfies the hypothesis of Theorem 2. 3, and hence $T = \sum_{i=0}^{m} \oplus z_i P_i$ where P_i is the projection of $\mathfrak{H}$ onto the null space of $T-z_i$, $i=0, 1, \ldots, m$. Since $|z_i|=1$ for $i=1, 2, \ldots, m$; $\sum_{i=1}^{m} \oplus z_i P_i$ is a unitary operator, and thus $T = U \oplus z_0 P_0$.

Corollary 2. 6. (FURUTA [2]) *If $T^k = T$ for some positive integer $k \geqq 2$ and $T \in C_\varrho$ ($\varrho>0$) then T is the direct sum of a zero operator and a unitary operator.*

Proof. Obvious from Theorem 2. 5.

§ 3. The class $\overline{\mathscr{R}}_1$

The class $\mathscr{R}_1$ of operators was introduced by HALMOS [3]. An operator T is in $\mathscr{R}_1$ if and only if T has one dimensional reducing subspace. $\overline{\mathscr{R}}_1$ denotes the norm closure of $\mathscr{R}_1$. HALMOS [3] showed that every normal operator and every isometry is in $\overline{\mathscr{R}}_1$. STAMPFLI [11] showed that $T+\mathscr{K} \subset \overline{\mathscr{R}}_1$ whenever T is either hyponormal or a Toeplitz operator, where $\mathscr{K}$ denotes the ideal of compact operators. He also showed that if the spectral radius of T is equal to the norm of T then T is in $\overline{\mathscr{R}}_1$. We shall prove that $T+\mathscr{K} \subset \overline{\mathscr{R}}_1$ whenever T is a sequentially G_1 operator.

The following four results are well known and are stated here for easy reference.

Lemma 3. 1. *$T \in \overline{\mathscr{R}}_1$ if and only if there exists a sequence of unit vectors $x_n \in \mathfrak{H}$ such that $\|(T-\lambda)x_n\| \to 0$ and $\|(T^*-\bar{\lambda})x_n\| \to 0$ for some $\lambda \in \sigma(T)$.*

Lemma 3. 2. *Let $\lambda_0 \in \sigma(T)$ where $|\lambda_0| = \|T\|$. Then there exists a sequence of unit vectors $x_n \in \mathfrak{H}$ such that $\|(T-\lambda_0)x_n\| \to 0$ and $\|(T^*-\bar{\lambda}_0)x_n\| \to 0$. Thus $T \in \overline{\mathscr{R}}_1$.*

Lemma 3. 3. *If there exists a sequence of unit vectors $x_n \in \mathfrak{H}$ such that x_n converges weakly to 0 and $\|(T-\lambda)x_n\| \to 0$ and $\|(T^*-\bar{\lambda})x_n\| \to 0$, then $T+\mathscr{K} \subset \overline{\mathscr{R}}_1$.*

Lemma 3. 4. *If there exists a sequence of unit vectors $x_n \in \mathfrak{H}$ such that x_n converges weakly to x_0 and $\|Tx_n\| \to 0$ then $Tx_0 = 0$.*

For any operator T on a Hilbert space $\mathfrak{H}$, let $\gamma_T = \{\lambda \in \sigma(T)$: there exists $x \in \mathfrak{H}$, $x \neq 0$ such that $(T-\lambda)x = 0$ and $(T^*-\bar{\lambda})x = 0\}$. If γ_T is not empty then $T \in \overline{\mathscr{R}}_1$. Also if γ_T is an infinite set then it can be easily shown that $T+\mathscr{K} \subset \overline{\mathscr{R}}_1$. In order to prove our result, we need the following lemmas.

Lemma 3. 5. *Suppose there exists a sequence of unit vectors $x_n \in \mathfrak{H}$ and $\lambda_0 \notin \gamma_T$ such that $\|(T-\lambda_0)x_n\| \to 0$ and $\|(T^*-\bar{\lambda}_0)x_n\| \to 0$ then $T+\mathscr{K} \subset \overline{\mathscr{R}}_1$.*

Proof. Since $\{x_n\}$ is a bounded sequence, we assume, without loss of generality, that x_n converges weakly to x_0. Then by Lemma 3. 4, $(T-\lambda_0)x_0 = 0$ and $(T^*-\bar{\lambda}_0)x_0 = 0$. Since $\lambda_0 \notin \gamma_T$ therefore $x_0 = 0$. Thus by Lemma 3. 3 $T+\mathscr{K} \subset \overline{\mathscr{R}}_1$.

Lemma 3. 6. *Let T be sequentially G_1 and suppose that γ_T is a finite set such that $\gamma_T = \sigma(T)$. Then $T+\mathscr{K} \subset \overline{\mathscr{R}}_1$.*

Proof. Let $\sigma(T) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Since $\gamma_T = \sigma(T)$, each λ_i is an eigenvalue of T. Also T is sequentially G_1, therefore by Lemma 2. 1, each λ_i is a reducing eigenvalue. If for some i, $\eta(T-\lambda_i)$ which is equal to $\eta(T^*-\bar{\lambda}_i)$ is infinite dimensional, then obviously $T+\mathscr{K} \subset \overline{\mathscr{R}}_1$. Otherwise we have $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ where $\mathfrak{H}_1 = \sum_{i=1}^{n} \oplus \eta(T-\lambda_i)$ and $\mathfrak{H}_2 = \mathfrak{H} \ominus \mathfrak{H}_1$ is infinite dimensional. Since $\mathfrak{H}_1$ reduces T and $\sigma(T)$ is a finite set it is not hard to verify that $T|\mathfrak{H}_2 = T_2$ is a sequentially G_1 operator and γ_{T_2} is empty. Note that $T_2+\mathscr{K} \subset \overline{\mathscr{R}}_1$ implies $T+\mathscr{K} \subset \overline{\mathscr{R}}_1$ and thus this case will be considered in the proof of the next theorem.

Theorem 3. 7. *If T is a sequentially G_1 operator, then $T+\mathcal{K} \subset \overline{\mathcal{R}}_1$.*

Proof. In view of Lemma 3. 6 we only need to consider the case when γ_T is a finite set and $\gamma_T \neq \sigma(T)$.

Since T is sequentially G_1, for any $\lambda_0 \in \partial(\sigma(T) \setminus \gamma_T)$, there exists a sequence $\mu_n \in \varrho(T)$ such that $\|(T-\mu_n)^{-1}\| = \frac{1}{d(\mu_n)}$, and $\mu_n \to \lambda_0$. Since $\mu_n \to \lambda_0 \notin \gamma_T$, therefore for any μ_{m_0} such that $|\mu_{m_0} - \lambda_0| < \min\{|\lambda_0 - \alpha| : \alpha \in \gamma_T\}$, $d(\mu_{m_0}) = |\mu_{m_0} - \lambda_{m_0}|$ where $\lambda_{m_0} \in \sigma(T) \setminus \gamma_T$.

Thus $\|(T-\mu_{m_0})^{-1}\| = \frac{1}{d(\mu_{m_0})} = \frac{1}{|\lambda_{m_0} - \mu_{m_0}|}$, $\frac{1}{\lambda_{m_0} - \mu_{m_0}} \in \sigma((T-\mu_{m_0})^{-1})$.

Hence by Lemma 3. 2, there exists a sequence of unit vectors x_n such that

$$\|[(T-\mu_{m_0})^{-1} - (\lambda_{m_0} - \mu_{m_0})^{-1}]x_n\| \to 0 \text{ and } \|[(T^* - \bar{\mu}_{m_0})^{-1} - (\bar{\lambda}_{m_0} - \bar{\mu}_{m_0})^{-1}]x_n\| \to 0.$$

Hence by the first resolvent equation we get $\|(T-\lambda_{m_0})x_n\| \to 0$ and $\|(T^* - \bar{\lambda}_{m_0})x_n\| \to 0$. Also $\lambda_{m_0} \notin \gamma_T$. Thus by Lemma 3. 5, $T+\mathcal{K} \subset \overline{\mathcal{R}}_1$.

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