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Generalizations of the Hardy—Littlewood inequality. II

By JÓZSEF NÉMETH in Szeged

1. G. H. HARDY and J. E. LITTLEWOOD [2] proved the following

Theorem A. *Suppose that* $a_n \geqq 0$ $(n=1, 2, \ldots)$ *and that c is a real number. Set*

$$A_{m,n} = \sum_{\nu=m}^{n} a_\nu .$$

If $p>1$ *we have*

$$\sum_{n=1}^{\infty} n^{-c} A_{1,n}^{p} \leqq K \sum_{n=1}^{\infty} n^{-c} (n \cdot a_n)^p \quad \textit{with} \quad c>1, ^{*)} \tag{1}$$

$$\sum_{n=1}^{\infty} n^{-c} A_{n,\infty}^{p} \leqq K \sum_{n=1}^{\infty} n^{-c} (n \cdot a_n)^p \quad \textit{with} \quad c<1; \tag{2}$$

and if $0<p<1$ *we have*

$$\sum_{n=1}^{\infty} n^{-c} A_{1,n}^{p} \geqq K \sum_{n=1}^{\infty} n^{-c} (n \cdot a_n)^p \quad \textit{with} \quad c>1, \tag{3}$$

$$\sum_{n=1}^{\infty} n^{-c} A_{n,\infty}^{p} \geqq K \sum_{n=1}^{\infty} n^{-c} (n \cdot a_n)^p \quad \textit{with} \quad c<1. \tag{4}$$

The inequalities (1) and (2) were generalized by H. P. MULHOLLAND [4], moreover (3) and (4) by CHEN YUNG MING [1], replacing the function x^p in (1)—(4) by more general functions, notably they proved inequalities of the following type

$$\sum_{n=1}^{\infty} n^{-c} \Phi(A_{1,n}) \leqq K \sum_{n=1}^{\infty} n^{-c} \Phi(n a_n), \tag{5}$$

$$\sum_{n=1}^{\infty} n^{-c} \Psi(A_{1,n}) \geqq K \sum_{n=1}^{\infty} n^{-c} \Psi(n a_n) \tag{6}$$

under certain conditions on the functions $\Phi(x)$, $\Psi(x)$ and C.

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*) K denotes a positive absolute constant, not necessarily the same at each occurrence.

Theorem A was generalized in another direction by L. LEINDLER [3], who replaced in (1)—(4) the sequence $\{n^{-c}\}$ by an arbitrary sequence $\{\lambda_n\}$; for instance he proved the following inequality:

$$\sum_{n=1}^{\infty} \lambda_n A_{1,n}^p \leqq p^p \sum_{n=1}^{\infty} \lambda_n^{1-p} \left(\sum_{m=n}^{\infty} \lambda_m \right)^p a_n^p \tag{7}$$

with $p \geqq 1$ and $\lambda_n > 0$.

In the present paper we prove a theorem which contains all of these results.

2. We use the following notations:

a) $\Phi(x)$ $(x \geqq 0)$ denotes a non-negative function such that $\varphi(x) = \Phi(x)/x$ is increasing and, for some $k > 1$, $f(x) = \Phi(x)/x^k$ is decreasing.

b) $\Psi(x)$ $(x \geqq 0)$ denotes a non-negative function increasing to infinity such that $\varrho(x) = \Psi(x)/x$ is decreasing to zero, when x is increasing from zero to infinity.

c) $\Lambda_{m,n} = \sum_{i=m}^{n} \lambda_i \quad (1 \leqq m \leqq n \leqq \infty)$.

3. We prove the following

Theorem. *If $a_n \geqq 0$ and $\lambda_n > 0$ $(n = 1, 2, \ldots)$, then*

$$\sum_{n=1}^{\infty} \lambda_n \Phi(A_{1,n}) \leqq K_1 \sum_{n=1}^{\infty} \lambda_n \Phi\left(\frac{a_n}{\lambda_n} \Lambda_{n,\infty} \right), \tag{8}$$

and

$$\sum_{n=1}^{\infty} \lambda_n \Phi(A_{n,\infty}) \leqq K_2 \sum_{n=1}^{\infty} \lambda_n \Phi\left(\frac{a_n}{\lambda_n} \Lambda_{1,n} \right), \tag{9}$$

where K_1 and K_2 are constants depending on Φ; furthermore

$$\sum_{n=1}^{\infty} \lambda_n \Psi\left(\frac{a_n}{\lambda_n} \Lambda_{n,\infty} \right) \leqq C_1 \sum_{n=1}^{\infty} \lambda_n \Psi(A_{1,n}) \tag{10}$$

and

$$\sum_{n=1}^{\infty} \lambda_n \Psi\left(\frac{a_n}{\lambda_n} \Lambda_{1,n} \right) \leqq C_2 \sum_{n=1}^{\infty} \lambda_n \Psi(A_{n,\infty}), \tag{11}$$

where C_1 and C_2 are positive absolute constants.

4. We remark that this theorem implies LEINDLER's theorem [3] and several results of CHEN YUNG MING [1] and H. P. MULHOLLAND [4]; the method of proof of (10) and (11) is similar to that of LEINDLER's theorem.

5. We require the following lemmas:

Lemma 1. *If $\Phi(x)$ and $\varphi(x)$ are the functions defined above and $a_\nu \geqq 0$, then*

$$\Phi(A_{1,n}) \leqq K \sum_{\nu=1}^{n} a_\nu \varphi(A_{1,\nu}).$$

Lemma 2. *If* $\Phi(x)$ *and* $\varphi(x)$ *are the functions defined above, and* $a_\nu \geqq 0$, *then for every natural number* N

$$\Phi(A_{n,N}) \leqq K \sum_{\nu=n}^{N} a_\nu \varphi(A_{\nu,N}).$$

Lemma 3. *If* $b_n > 0$, $c_n \geqq 0$, $a_n \geqq 0$ $(n = 1, 2, \ldots)$ *and if for every natural number* N

$$\sum_{n=1}^{N} b_n \Phi(A_{n,N}) \leqq K \sum_{n=1}^{N} c_n,$$

then

$$\sum_{n=1}^{\infty} b_n \Phi(A_{n,\infty}) \leqq K \sum_{n=1}^{\infty} c_n.$$

6. Proof of Lemma 1. Let $f(x)$ be the function defined above, in point 2, and write A_n instead of $A_{1,n}$. Then

$$\Phi(A_n) = \Phi(A_1) + \Phi(A_2) - \Phi(A_1) + \cdots + \Phi(A_n) - \Phi(A_{n-1});$$

as

$$\Phi(A_m) - \Phi(A_{m-1}) = f(A_m)A_m^k - f(A_{m-1})A_{m-1}^k \leqq$$

$$\leqq kf(A_m)A_m^{k-1}(A_m - A_{m-1}) = k\varphi(A_m)a_m \quad \text{for} \quad m \geqq 2,$$

and $\Phi(A_1) \leqq k\varphi(A_1)a_1$, we obtain the assertion.

Proof of Lemma 2. Let us write B_n for $A_{n,N}$ $(N \geqq n)$. Then

$$\Phi(B_n) = \Phi(B_n) - \Phi(B_{n+1}) + \cdots + \Phi(B_{N-1}) - \Phi(B_N) + \Phi(B_N);$$

using the estimations

$$\Phi(B_m) - \Phi(B_{m+1}) = f(B_m)B_m^k - f(B_{m+1})B_{m+1}^k \leqq k\varphi(B_m)(B_m - B_{m+1}) = k\varphi(B_m)a_m,$$

and

$$\Phi(B_N) \leqq k\varphi(B_N)a_N,$$

we obtain the assertion.

Proof of Lemma 3. This can be done by an easy computation.

7. Proof of the Theorem. *Inequality* (8). Applying Lemma 1 we obtain that

$$\sum_{n=1}^{N} \lambda_n \Phi(A_{1,n}) \leqq k \sum_{n=1}^{N} \lambda_n \sum_{\nu=1}^{n} a_\nu \varphi(A_{1,\nu}) = \sum\nolimits_1$$

holds for every natural number N.

Interchanging the order of the summations we have

$$\sum\nolimits_1 = k\sum_{\nu=1}^{N} a_\nu \varphi(A_{1,\nu})\Lambda_{\nu,N} = k\sum_{\nu=1}^{N} t^{-1}\left\{t\frac{a_\nu}{\lambda_\nu}\Lambda_{\nu,N}\varphi(A_{1,\nu})\lambda_\nu\right\}.$$

Since

$$x\varphi(y) \leqq x\varphi(x)+y\varphi(y) = \Phi(x)+\Phi(y) \quad \text{for } x\geqq 0,\ y\geqq 0 \tag{12}$$

and

$$\Phi(tx) = f(tx)t^k x^k \leqq t^k f(x)x^k = t^k\Phi(x) \quad \text{for } t\geqq 1,\ x\geqq 0, \tag{13}$$

we obtain

$$\sum\nolimits_1 \leqq k\sum_{\nu=1}^{N} t^{-1}\left\{\Phi\left(t\frac{a_\nu}{\lambda_\nu}\Lambda_{\nu,N}\right)\lambda_\nu + \Phi(A_{1,\nu})\lambda_\nu\right\} \leqq \tag{14}$$

$$\leqq k\sum_{\nu=1}^{N}\left\{t^{k-1}\Phi\left(\frac{a_\nu}{\lambda_\nu}\Lambda_{\nu,N}\right)\lambda_\nu + t^{-1}\Phi(A_{1,\nu})\lambda_\nu\right\}.$$

Hence, choosing t such that $1-kt^{-1}$ be positive, we obtain

$$\sum_{n=1}^{N}\lambda_n\Phi(A_{1,n}) \leqq \frac{k\cdot t^{k-1}}{1-k\cdot t^{-1}}\sum_{n=1}^{N}\lambda_n\Phi\left(\frac{a_n}{\lambda_n}\Lambda_{n,N}\right),$$

which proves (8).

Inequality (9). Applying Lemma 2 we have, for an arbitrary natural number N,

$$\sum_{n=1}^{N}\lambda_n\Phi(A_{n,N}) \leqq k\sum_{n=1}^{N}\lambda_n\sum_{\nu=n}^{N} a_\nu\varphi(A_{\nu,N}) = \sum\nolimits_2.$$

Interchanging the order of the summations we obtain

$$\sum\nolimits_2 = k\sum_{\nu=1}^{N} a_\nu\varphi(A_{\nu,N})\Lambda_{1,n} = k\sum_{\nu=1}^{N} t^{-1}\left\{t\frac{a_\nu}{\lambda_\nu}\Lambda_{1,n}\varphi(A_{\nu,N})\lambda_\nu\right\}.$$

Using (12) and (13), similarly to (14), we have

$$\sum\nolimits_2 \leqq k\sum_{\nu=1}^{N}\left\{t^{k-1}\Phi\left(\frac{a_\nu}{\lambda_\nu}\Lambda_{1,\nu}\right)\lambda_\nu + t^{-1}\Phi(A_{\nu,N})\lambda_\nu\right\}.$$

Hence, if $1-kt^{-1} > 0$ we get

$$\sum_{n=1}^{N}\lambda_n\Phi(A_{n,N}) \leqq \frac{kt^{k-1}}{1-kt^{-1}}\sum_{n=1}^{N}\lambda_n\Phi\left(\frac{a_n}{\lambda_n}\Lambda_{1,n}\right),$$

and using Lemma 3, we obtain (9).

Inequality (10). We may suppose that the series on the right-hand side is convergent, and thus we can define a sequence $\{m_n\}$ in the following way:

Let $m_0=0$ and for $n\geqq 1$ let m_n be the smallest natural number $k(>m_{n-1})$ such that $\Lambda_{m_{n-1}+1,k}\geqq\Lambda_{k+1,\infty}$. Then $\Lambda_{m_n+1,m_{n+1}}\geqq\Lambda_{m_{n+1}+1,\infty}$, and

$$\Lambda_{m_n+1,m_{n+1}}\leqq 2\Lambda_{m_{n+1},m_{n+2}}, \tag{15}$$

and

$$\Lambda_{m_n+1,\infty}\leqq 2\Lambda_{m_n+1,m_{n+1}}. \tag{16}$$

We first show that

$$\sum_{\nu=m_n+1}^{m_{n+1}}\lambda_\nu\Psi\left(\Lambda_{m_n+1,\infty}\cdot\frac{a_\nu}{\lambda_\nu}\right)\leqq 2\Psi(A_{m_n+1,m_{n+1}})\Lambda_{m_n+1,\infty}. \tag{17}$$

We use the following notations:

$$\tau_\nu^{(n)}=\frac{\Lambda_{m_n+1,m_{n+1}}}{\lambda_\nu}$$

Using the properties of the functions $\Psi(x)$, $\varrho(x)$ we get:

$$\sum_{\nu=m_n+1}^{m_{n+1}}\lambda_\nu\Psi\left(\Lambda_{m_n+1,\infty}\cdot\frac{a_\nu}{\lambda_\nu}\right)\leqq\Lambda_{m_n+1,\infty}\sum_{\nu=m_n+1}^{m_n+1}a_\nu\varrho(\tau_\nu^{(n)}a_\nu). \tag{18}$$

Let ν_i, $\bar\nu_j$ denote the subscripts such that $m_n+1\leqq\nu_i,\ \bar\nu_j\leqq m_{n+1}$, and

$$\tau_{\nu_i}^{(n)}a_{\nu_i}\leqq A_{m_n+1,m_{n+1}},\qquad \tau_{\bar\nu_j}^{(n)}a_{\bar\nu_j}>A_{m_n+1,m_{n+1}}.$$

Then

$$\sum_{\nu=m_n+1}^{m_{n+1}}a_\nu\varrho(\tau_\nu^{(n)}a_\nu)=\sum_{\nu=m_n+1}^{m_{n+1}}\frac{a_\nu\tau_\nu^{(n)}\varrho(\tau_\nu^{(n)}a_\nu)}{\tau_\nu^{(n)}}\leqq\sum_i^{(1)}\frac{A_{m_n+1,m_{n+1}}\varrho(A_{m_n+1,m_{n+1}})}{\tau_{\nu_i}^{(n)}}+$$

$$+\sum_j^{(2)}\varrho(A_{m_n+1,m_{n+1}})a_{\bar\nu_j}\leqq A_{m_n+1,m_{n+1}}\varrho(A_{m_n+1,m_{n+1}})\sum_i^{(1)}\frac{1}{\tau_{\nu_i}^{(n)}}+$$

$$+\varrho(A_{m_n+1,m_{n+1}})\sum_j^{(2)}a_{\bar\nu_j}\leqq 2\Psi(A_{m_n+1,m_{n+1}}).$$

thus, by (18), we get (17). Using (15), (16) and (17) we have:

$$\sum_{\nu=m_n+1}^{m_{n+1}}\lambda_\nu\Psi\left(\Lambda_{\nu,\infty}\frac{a_\nu}{\lambda_\nu}\right)\leqq 2\Lambda_{m_n+1,\infty}\Psi(A_{m_n+1,m_{n+1}})\leqq$$

$$\leqq 2\Lambda_{m_n+1,\infty}\Psi(A_{1,m_{n+1}})\leqq 4\Lambda_{m_n+1,m_{n+1}}\Psi(A_{1,m_{n+1}})\leqq 8\sum_{k=m_{n+1}}^{m_{n+2}}\lambda_k\Psi\left(\sum_{\nu=1}^{k}a_\nu\right).$$

Hence

$$\sum_{\nu=1}^{\infty} \lambda_\nu \Psi\left(\Lambda_{\nu,\infty} \frac{a_\nu}{\lambda_\nu}\right) \leqq 8 \sum_{n=0}^{\infty} \sum_{\nu=m_{n+1}}^{m_{n+2}} \lambda_\nu \Psi\left(\sum_{k=1}^{\nu} a_k\right) \leqq 16 \sum_{\nu=1}^{\infty} \lambda_\nu \Psi\left(\sum_{k=1}^{\nu} a_k\right),$$

which gives (10).

Inequality (11). We distinguish two cases. First we suppose

$$\Lambda_{1,\infty} < \infty.$$

We define a sequence of integers $\mu_0, \mu_1, \ldots$. We set $\mu_0=0$, $\mu_1=1$ and if μ_n has already been defined we choose $\mu'_{n+1}=k$, where $k(>\mu_n)$ denotes the smallest integer satisfying

$$(19) \qquad \Lambda_{\mu_n+1,k} \geqq 3\Lambda_{\mu_{n-1}+1,\mu_n}$$

provided such a k exists. If $\mu'_{n+1} > \mu_n+1$ then let $\mu_{n+1} = \mu'_{n+1}-1$ and if $\mu'_{n+1} = \mu_n+1$ then let $\mu_{n+1} = \mu_n+1$. If there exists no natural number k with (19) then let $\mu_{n+1}=\infty$. It is clear that this inductive definition always stops at some $n=N_0$, that is $\mu_{N_0}=\infty$ holds. For in opposite case, by the definition of μ_n, the inequality

$$(20) \qquad 3I_{n-2} \leqq I_{n-1}+I_n$$

holds for all $2 \leqq n < N_0$, where $I_n = \Lambda_{\mu_n+1, \mu_{n+1}}$ and inequality (20) for infinitely many n would imply $\Sigma\lambda_k=\infty$ contrary to the assumption. By (20) we have for $1 \leqq n < N_0-1$

$$(21) \qquad \Lambda_{1,\mu_n} \leqq 3I_{n-1}+I_n$$

and

$$(22) \qquad \Lambda_{1,\mu_{N_0}} \leqq 3I_{N_0-3}+5I_{N_0-2}.$$

Next we remark that

$$(23) \qquad \sum_{\nu=\mu_n+1}^{\mu_{n+1}} \lambda_\nu \Psi\left(\Lambda_{1,\nu} \frac{a_\nu}{\lambda_\nu}\right) \leqq 2\Lambda_{1,\mu_{n+1}} \Psi(A_{\mu_n+1,\mu_{n+1}}).$$

By the properties of the functions $\Psi(x)$, $\varrho(x)$ we have

$$\sum_{\nu=\mu_n+1}^{\mu_{n+1}} \lambda_\nu \Psi\left(\Lambda_{1,\nu} \frac{a_\nu}{\lambda_\nu}\right) \leqq \sum_{\nu=\mu_n+1}^{\mu_{n+1}} \lambda_\nu \Psi\left(\Lambda_{1,\mu_{n+1}} \frac{a_\nu}{\lambda_\nu}\right) =$$

$$= \Lambda_{1,\mu_{n+1}} \sum_{\nu=\mu_n+1}^{\mu_{n+1}} a_\nu \varrho\left(\Lambda_{1,\mu_{n+1}} \frac{a_\nu}{\lambda_\nu}\right) \leqq \Lambda_{1,\mu_{n+1}} \sum_{\nu=\mu_n+1}^{\mu_{n+1}} a_\nu \varrho\left(\Lambda_{\mu_n+1,\mu_{n+1}} \frac{a_\nu}{\lambda_\nu}\right).$$

Hence, applying the idea of proof of (17), we obtain (23) immediately.

Using (23) we get

$$\sum_{n=0}^{N_0-1}\sum_{\nu=\mu_n+1}^{\mu_{n+1}}\lambda_\nu\Psi\left(\Lambda_{1,\nu}\frac{a_\nu}{\lambda_\nu}\right)\leqq 2\sum_{n=0}^{N_0-1}\Lambda_{1,\mu_{n+1}}\Psi(A_{\mu_n+1,\mu_{n+1}})=\sum\nolimits_3. \tag{24}$$

By the definition of the sequence $\{\mu_n\}$ and by (21) we have

$$\sum\nolimits_3\leqq 2\sum_{n=0}^{1}\Lambda_{1,\mu_{n+1}}\Psi(A_{\mu_n+1,\mu_{n+1}})+2\sum_{n=2}^{N_0-2}\Lambda_{1,\mu_{n+1}}\Psi(A_{\mu_n+1,\mu_{n+1}})+ \tag{25}$$

$$+2\Lambda_{1,\mu_{N_0}}\Psi(A_{\mu_{N_0-1}+1,\infty}) = \sum\nolimits_4+\sum\nolimits_5+\sum\nolimits_6.$$

Using (19) and (21) we get

$$\sum\nolimits_5\leqq 2\sum_{n=2}^{N_0-2}(\Lambda_{1,\mu_{n-1}}+\Lambda_{\mu_{n-1}+1,\mu_{n+1}})\Psi(A_{\mu_n+1,\mu_{n+1}})\leqq \tag{26}$$

$$\leqq 2\sum_{n=2}^{N_0-2}(3\Lambda_{\mu_{n-2}+1,\mu_{n-1}}+2\Lambda_{\mu_{n-1}+1,\mu_n}+\Lambda_{\mu_n+1,\mu_{n+1}})\Psi(A_{\mu_n+1,\mu_{n+1}})\leqq$$

$$\leqq 2\sum_{n=2}^{N_0-2}(3\Lambda_{\mu_{n-2}+1,\mu_{n-1}}+5\Lambda_{\mu_{n-1}+1,\mu_n}+\lambda_{\mu_n+1})\Psi(A_{\mu_n+1,\mu_{n+1}}).$$

An easy computation gives by (19) and (21) that

$$\sum\nolimits_4 \leqq 2[5\lambda_1\Psi(A_{1,\infty})+\lambda_2\Psi(A_{2,\infty})]. \tag{27}$$

By (22) we obtain

$$\Lambda_{1,\mu_{N_0}}\Psi(A_{\mu_{N_0-1}+1,\infty})\leqq(3\Lambda_{\mu_{N_0-3}+1,\mu_{N_0-2}}+5\Lambda_{\mu_{N_0-2}+1,\mu_{N_0-1}})\Psi(A_{\mu_{N_0-1}+1,\infty}). \tag{28}$$

Using (26), (27), (28) we get

$$\sum\nolimits_4+\sum\nolimits_5+\sum\nolimits_6 \leqq 18\sum_{n=1}^{\infty}\lambda_n\Psi\left(\sum_{k=n}^{\infty}a_k\right),$$

which by (24) and (25) gives (11) in case $\Sigma\lambda_k<\infty$.

If $\Sigma\lambda_n=\infty$ then we define another index-sequence $\{m_n\}$. Let $m_0=0$ and $m_1=1$. If $m_0<m_1<\dots<m_n$ $(n\geqq 1)$ have been defined, then let m_{n+1} be the smallest natural number k with

$$\Lambda_{m_n+1,k}\geqq 2\Lambda_{m_{n-1}+1,m_n} \tag{29}$$

By the definition of m_n we have

$$\Lambda_{1,m_{n+1}-1} \leqq \Lambda_{m_{n+1},m_{n+1}-1}+2\Lambda_{m_{n-1},m_n}, \tag{30}$$

$$\Lambda_{1,m_2-1}\leqq 3\lambda_1, \tag{31}$$

$$\Lambda_{m_{n-1},m_{n+1}-1}\leqq 3\Lambda_{m_{n-1},m_n}. \tag{32}$$

First we remark that similarly to the proof of (17) and (23) we obtain the following inequalities

$$\sum_{k=m_n}^{m_{n+1}-1} \lambda_k \cdot \Psi\left(\Lambda_{m_{n-1}, m_{n+1}-1} \frac{a_k}{\lambda_k}\right) \leqq 2\Lambda_{m_{n-1}, m_{n+1}-1} \Psi(A_{m_n, m_{n+1}-1}), \tag{33}$$

$$\sum_{k=1}^{m_2-1} \lambda_k \Psi\left(\Lambda_{1, m_2-1} \frac{a_k}{\lambda_k}\right) \leqq 2\Lambda_{1, m_2-1} \Psi(A_{1, m_2-1}). \tag{34}$$

By the definition of sequence $\{m_n\}$ and by (30) we have

$$\sum_{k=1}^{\infty} \lambda_k \Psi\left(\Lambda_{1,k} \frac{a_k}{\lambda_k}\right) \leqq \sum_{n=1}^{\infty} \sum_{k=m_n}^{m_{n+1}-1} \lambda_k \Psi\left(\Lambda_{1, m_{n+1}-1} \frac{a_k}{\lambda_k}\right) \leqq$$

$$\leqq \sum_{k=1}^{m_2-1} \lambda_k \Psi\left(\Lambda_{1, m_2-1} \frac{a_k}{\lambda_k}\right) + \sum_{n=2}^{\infty} \sum_{k=m_n}^{m_{n+1}-1} \lambda_k \Psi\left[\frac{a_k}{\lambda_k}(\Lambda_{m_n+1, m_{n+1}-1} + 2\Lambda_{m_{n-1}, m_n})\right] = \sum\nolimits_7.$$

Since $\Psi(2x) \leqq 2\Psi(x)$,

$$\sum\nolimits_7 \leqq \sum_{k=1}^{m_2-1} \lambda_k \Psi\left(\Lambda_{1, m_2-1} \frac{a_k}{\lambda_k}\right) + 2 \sum_{n=2}^{\infty} \sum_{k=m_n}^{m_{n+1}-1} \lambda_k \Psi\left(\Lambda_{m_{n-1}, m_{n+1}-1} \frac{a_k}{\lambda_k}\right) = \sum\nolimits_8.$$

By (33) and (34),

$$\sum\nolimits_8 \leqq 2\Lambda_{1, m_2-1} \Psi(A_{1, m_2-1}) + 4 \sum_{n=2}^{\infty} \Lambda_{m_{n-1}, m_{n+1}-1} \Psi(A_{m_n, m_{n+1}-1}) = \sum\nolimits_9.$$

Using (31), (32) we get

$$\sum\nolimits_9 \leqq 6\lambda_1 \Psi(A_{1,\infty}) + 12 \sum_{n=2}^{\infty} \Lambda_{m_{n-1}, m_n} \Psi(A_{m_n, \infty}) \leqq 24 \sum_{n=1}^{\infty} \lambda_n \Psi\left(\sum_{\nu=n}^{\infty} a_\nu\right),$$

which is the required inequality (11). The proof is complete.

References

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