A generalization of the Halmos—Bram criterion for subnormality

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Introduction. In [3] and [1] HALMOS and BRAM show that a continuous linear operator A on a complex Hilbert space X is subnormal if and only if $\sum_{i,j=0}^{n} (A^i x_j, A^j x_i) \geqq 0$ for all finite collections $x_0, \dots, x_n$ of X. In Section 1 we generalize this criterion by showing that A is subnormal if and only if $\sum_{i,j=0}^{n} (A^{i+j} x_j, A^{i+j} x_i) \geqq 0$ for all finite subcollections $x_0, \dots, x_n$ of X. As an application of this criterion we show in Section 2 that an operator A is the restriction of a normal partial isometry to an invariant subspace if and only if $A = A^* A^2$ and $\|A\| \leqq 1$. In Section 3 we show, using our new criterion for subnormality, that an operator A is subnormal if and only if $\{A^{*n} A^n\}_{n=0}^{\infty}$ is a Hausdorff moment sequence.

Throughout the paper X is a complex Hilbert space with inner product $(\, , \,)$ and norm $\| \; \|$. If B is a continuous linear operator on X, then B^* is the adjoint of B. B is *normal* if $BB^* = B^* B$, *quasi-normal* if $B(B^* B) = (B^* B)B$, an *isometry* if $B^* B = I$ and a *partial isometry* if $(B^* B)^2 = B^* B$. An operator A is *subnormal* if it is the restriction of a normal operator B to an invariant subspace of B and *hyponormal* if $AA^* \leqq A^* A$. A sequence $\{C_n\}_0^\infty$ of operators on X is a *Hausdorff moment sequence* if there exists a positive operator measure φ on some interval $[a, b]$ such that $C_n = \int_a^b t^n \, d\varphi$ for each nonnegative integer n.

1. A criterion for subnormality. The Halmos—Bram criterion that an operator A on X be subnormal is that $\sum_{i,j=0}^{n} (A^i x_j, A^j x_i) \geqq 0$ for all finite collections $x_0, \dots, x_n$ of X. We generalize this as follows:

Theorem 1. *An operator A on a complex Hilbert space X is subnormal if and only if A satisfies*

$$(\mathrm{S}_1) \qquad \sum_{i,j=0}^{n} (A^{i+j} x_j, A^{i+j} x_i) \geqq 0$$

for each finite collection $x_0, \dots, x_n$ of X.

Proof. The proof of the necessity of the condition is easy. Note that (S_1) is the special case of the Halmos—Bram condition which we obtain by choosing $x_i = A^i x_i'$ for $i=0, \ldots, n$.

To prove the sufficiency of the condition we imitate the techniques of Halmos and Bram and prove that if A satisfies condition (S_1), then A is the restriction of a quasi-normal operator to an invariant subspace. This will complete our proof, since every quasi-normal operator is subnormal ([4, problem 154]).

Assume now that A satisfies condition (S_1). The first step in the proof will be to show that A also satisfies

$$(S_2) \qquad \sum_{i,j=0}^{n} (A^{i+j+1}x_j, A^{i+j+1}x_i) \leqq \|A\|^2 \sum_{i,j=0}^{n} (A^{i+j}x_j, A^{i+j}x_i).$$

To obtain this result we outline a modification of BRAM's proof in [1, Theorem 1, p. 76].

Let $\varepsilon > 0$ and let $A_1 = A/(\|A\| + \varepsilon)$. A_1 also satisfies condition (S_1). Let $Y = l^2(X)$. Define C on Y by $(Cy)_i = \sum_{j=0}^{\infty} A_1^{*i+j} A_1^{i+j} y_j$. An argument similar to that used by Bram shows that C is a well-defined, bounded operator on Y and that $C \geqq 0$ on Y. Now define B on Y by $(By)_i = A_1 y_i$. A computation almost identical to that used by Bram shows that $\|B^* CBy\| \leqq \|Cy\|$ for all y in Y and hence by [5, p. 426] that $B^* CB \leqq C$ since $\|B\| = \|A_1\| < 1$. It now follows that if $x_0, \ldots, x_n$ are elements of X, then

$$\sum_{i,j=0}^{n} (A^{i+j+1}x_j, A^{i+j+1}x_i) \leqq (\|A\| + \varepsilon)^2 \sum_{i,j=0}^{n} (A^{i+j}x_j, A^{i+j}x_i).$$

Since ε was an arbitrary positive number, condition (S_2) is satisfied.

The second step in the proof of the theorem is the construction of a quasi-normal extension of A. The following modification of HALMOS' proof in [3] will give us this result.

Let $\tilde{X}$ be the set of all sequences $\{x_i\}_{i=-\infty}^{\infty}$ in X such that $x_i = 0$ for $i < 0$ and $x_i \neq 0$ for at most a finite number of i. On $\tilde{X}$ define

$$(\tilde{x}, \tilde{y}) = \sum_{i,j} (A^{i+j}x_j, A^{i+j}y_i).$$

Let $\tilde{Y}$ be the set of equivalence classes obtained by identifying $\tilde{x}$ with 0 if $(\tilde{x}, \tilde{x}) = 0$. Then since A satisfies condition (S_1), $(\ ,\ )$ is an inner product on $\tilde{Y}$. Define D on

$\tilde{X}$ by $(D\tilde{x})_i = Ax_i$. Using the fact that A also satisfies condition (S_2), we have

$$(D\tilde{x}, D\tilde{x}) = \sum_{i,j} (A^{i+j+1}x_i, A^{i+j+1}x_j) \leqq \|A\|^2 \sum_{i,j} (A^{i+j}x_i, A^{i+j}x_j) = \|A\|^2 (\tilde{x}, \tilde{x}).$$

It now follows that D may be considered to be a continuous linear operator on $\tilde{Y}$.

Define E on $\tilde{X}$ by $(E\tilde{x})_i = x_{i-1}$ and note that $DE = ED$ on $\tilde{X}$. Furthermore on $\tilde{X}$ we have the relation

$$(D\tilde{x}, D\tilde{y}) = \sum_{i,j} (A^{i+j+1}x_i, A^{i+j+1}x_j) = \sum_{i,j} (A^{i+j}x_i, A^{i+j}x_{j-1}) = (\tilde{x}, E\tilde{y}).$$

Thus on the completion of $\tilde{Y}$, the extensions of D and E satisfy the equation $E = D^* D$. However, we have already observed that D commutes with E. Therefore the extension of D to the completion of $\tilde{Y}$ is a quasi-normal extension of A and the proof of the theorem is complete.

In [6, Theorem 7, p. 73] Mac Nerney shows that a sequence $\{C_n\}_{n=0}^{\infty}$ of Hermitian operators on X is a Hausdorff sequence for the interval $[a, b]$ if and only if

$$a \sum_{i,j=0}^{n} (x_i, C_{i+j}x_j) \leqq \sum_{i,j=0}^{n} (x_i, C_{i+j+1}x_j) \leqq b \sum_{i,j=0}^{n} (x_i, C_{i+j}x_j)$$

for each finite collection $x_0, \ldots, x_n$ in X. Using this result and Theorem 1, we readily obtain the following:

Corollary. *An operator A on X is subnormal if and only if $\{A^{*n}A^n\}_{n=0}^{\infty}$ is a Hausdorff moment sequence.*

We note that if A is subnormal, then A is quasi-normal if and only if $A^{*n}A^n = \int t^n d\varphi$ where φ is a spectral measure (that is, φ is a projection-valued operator measure). The proof of this assertion is simple. If A is quasi-normal, then $A^{*n}A^n = (A^*A)^n$ for $n \geqq 0$ and thus $A^{*n}A^n = \int t^n d\varphi$ where φ is the spectral resolution of A^*A. Conversely, if $A^{*n}A^n = \int t^n d\varphi$ and φ is projection-valued, then $A^{*n}A^n = (A^*A)^n$ for $n \geqq 0$. Furthermore, by the last corollary A is subnormal and hence hyponormal. However if A is hyponormal and $A^{*2}A^2 = (A^*A)^2$, then $(A^*A - AA^*)A = 0$, proving that A is quasi-normal.

2. The operator equation $A = A^*A^2$. Consider the weighted shift A on l^2 defined by $A(x_0, x_1, \ldots) = (0, 2x_0, x_1, x_2, \ldots)$. A simple computation shows that $A = A^*A^2$. However, since the weights of A are not monotone increasing, A is not hyponormal [4, p. 160] and consequently not subnormal. Thus not every operator satisfying the equation $A = A^*A^2$ is subnormal. The additional hypothesis needed to force A to be subnormal is that $\|A\| \leqq 1$. We are now able to completely characterize operators satisfying these two conditions.

Theorem 2. *Let A be an operator on a complex Hilbert space X. A is subnormal and the minimal normal extension of A is a partial isometry if and only if $\|A\| \leqq 1$ and $A = A^*A^2$.*

Proof. Assume first that B is a normal partial isometry on a Hilbert space Y, containing X, and that $B = A$ on X. Since every partial isometry has norm $\leqq 1$, we have $\|A\| \leqq \|B\| \leqq 1$. Let P be the projection of Y onto X. Then for each x in X we have $A^*A^2x = PB^*B^2x = PBB^*Bx$ (since B is normal) $= PBx$ (since B is a partial isometry) $= Bx = Ax$ and consequently, $A = A^*A^2$.

Now assume that $A = A^*A^2$ and $\|A\| \leqq 1$. A simple inductive argument shows that $A^{*k}A^k = A^*A$ for each integer $k \geqq 1$. Therefore if $x_0, \ldots, x_n$ are elements of X,

$$\sum_{i,j=0}^{n} (A^{i+j}x_j, A^{i+j}x_i) = \sum_{i,j=0}^{n} (Ax_j, Ax_i) + \|x_0\|^2 - \|Ax_0\|^2 =$$

$$= \left\| \sum_{i=0}^{n} Ax_i \right\|^2 + \|x_0\|^2 - \|Ax_0\|^2 \geqq 0 \text{ since } \|A\| \leqq 1.$$

By Theorem 1 we know that A is subnormal. Let $B: Y \to Y$ be the minimal normal extension of A. It remains to show that B is a partial isometry. Let P be the projection of Y onto X. Then for x in X, $\|PB^*B^2x\| = \|A^*A^2x\| = \|Ax\| = \|A^3x\|$ (since $A^{*3}A^3 = A^*A$) $= \|B^3x\| = \|B^*B^2x\|$. Therefore $B^*B^2x \in X$ for all x and consequently $B^*B^2 = B$ on X. Since B is the minimal normal extension of A, the set of vectors $\left\{ \sum_{i=0}^{n} B^{*i}x_i : x_i \in X \right\}$ is dense in Y and consequently $B^*B^2 = B$ on a dense subset of Y. This is sufficient to imply that B is a partial isometry. The proof is complete.

The assertion in Theorem 2 parallels the assertion that an operator A is an isometry if and only if A is subnormal and the minimal normal extension of A is unitary.

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