A weakening of the definition of C*-algebras

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Dedicated to Professor Béla Sz.-Nagy on his 60th birthday

In a recent paper H. ARAKI and G. A. ELLIOTT proved the following theorem (see [1], Theorem 1):

Let A be a complex involutory algebra with complete linear space norm such that

(1) $$\|x^* \cdot x\| = \|x\|^2 \quad \text{for all } x \in A.$$

A is then a C^*-algebra, i.e. the submultiplicativity property

(2) $$\|x \cdot y\| \leqq \|x\| \cdot \|y\|$$

also holds for every $x, y \in A$.

These authors raised the problem whether it is enough to assume (1) for normal x only, i.e. for which $x^*x = xx^*$.

The answer is in the negative as was shown in [5] by the simple counter example of the algebra A of all bounded linear operators on a complex Hilbert space with the numerical radius as norm. This norm does not satisfy

(3) $$\|x^*x\| \leqq \|x\|^2 \quad \text{for every} \quad x \in A.$$

The purpose of this note is to prove that (3), together with (1) only for normal $x \in A$, is sufficient for A to be a C^*-algebra.

We shall use the notation of RICKART's book [4]. The following lemma, similar to Lemma 1 in [5], plays an important role in the arguments. Denote by $H(A)$ the selfadjoint part of A.

Lemma 1. *Let A be a complex involutory algebra with linear space norm which satisfies* (3). *Then A is a normed algebra with continuous involution.*

Proof. The first step is to prove

(4) $$\|hk\| \leqq 4\|h\|\|k\| \quad \text{for every} \quad h, k \in H(A).$$

Consider for $h, k \in H(A)$ the identity

$$4hk = (h+k)^2 - (h-k)^2 + i(h+ik)(h-ik) - i(h-ik)(h+ik)$$

which is a special case of (3) in [1]. Use the triangle inequality together with (3) to have thus $\|hk\| \leqq (\|h\|+\|k\|)^2$. Assume that h and k differ from 0, otherwise (4) is immediate, and replace them by $h/\|h\|$ and $k/\|k\|$, respectively; then (4) follows immediately.

We define an auxiliary linear space norm as follows: for $h, k \in H(A)$ let

$$\|h+ik\|_1 = \frac{1}{\sqrt{2}} \sup \{\|h \cdot \cos t - k \cdot \sin t\| + \|h \cdot \sin t + k \cdot \cos t\| : t \text{ real number}\}$$

so that

$$\frac{1}{\sqrt{2}}(\|h\|+\|k\|) \leqq \|h+ik\|_1 \leqq \|h\|+\|k\|$$

holds (for details see [4], p. 7). Moreover, the 1-norm agrees with the original norm on $H(A)$ and the involution is an isometry with this norm. The multiplication is also continuous with the 1-norm as for all $x, y \in A$ the inequality

$$\|xy\|_1 \leqq 8\|x\|_1 \cdot \|y\|_1$$

holds. It follows that the norm of the extended left regular representation on A with 1-norm, defined for $x \in A$ by

$$\|x\|_2 = \sup \{\|\lambda x + xy\|_1 : \lambda \text{ complex number}, \quad y \in A; \quad |\lambda| + \|y\|_1 = 1\},$$

is an appropriate norm. Indeed, it is equivalent to the 1-norm, as it is not hard to see that

$$\|x\|_1 \leqq \|x\|_2 \leqq 8\|x\|_1$$

for any $x \in A$, so that the involution is a norm-continuous map with the 2-norm. This completes the proof.

In the following $\nu(x)$ denotes the spectral radius of $x \in A$ with respect to the 2-norm

$$\nu(x) = \lim \|x^n\|_2^{1/n}.$$

The next result is not an evident consequence of the Araki—Elliott theorem mentioned earlier, but it follows from Lemma 1 by the properties of the spectral radius.

Proposition 2. *Let A be a complex commutative involutory algebra with linear space norm such that* (1) *holds for any $x \in A$. Then A is a pre-C^*-algebra.*

Proof. We show first

(5) $$\nu(h) = \|h\| \quad \text{for every} \quad h \in H(A)$$

by (1) and the equivalence of the norms on $H(A)$ as follows:

$$\nu(h) = \lim_{n\to\infty} \|h^{2^n}\|_2^{2^{-n}} = \lim_{n\to\infty} \|h^{2^n}\|_1^{2^{-n}} = \lim_{n\to\infty} \|h^{2^n}\|^{2^-} = \|h\|,$$

where in the last step the immediate consequence of (1)

$$\|h \quad\| = \|h\|^{2^n} \qquad (n=1, 2, \ldots; \quad h \in H(A))$$

was used. The only required property of the original norm follows from (5) by (1):

$$\|xy\| = \|y^*x^*xy\|^{1/2} = \nu(x^*xy^*y)^{1/2} \leqq \nu(x^*x)^{1/2}\nu(y^*y)^{1/2} =$$

$$= \|x^*x\|^{1/2}\|y^*y\|^{1/2} = \|x\|\,\|y\|$$

holds for any $x, y \in A$. Thus A is a pre-C^*-algebra with the original norm in fact.

The main result of this paper is the following

Theorem 3. *Let A be a complex involutory algebra with complete linear space norm which satisfies* (3) *and for which* (1) *holds for every normal $x \in A$. Then A is a C^*-algebra.*

Proof. Proposition 2 implies that every maximal commutative selfadjoint subalgebra of A is a pre-C^*-algebra. Consider now $A^\sim$, the norm completion of A in the 2-norm with the extended involution. We shall show that $A^\sim$ is a C^*-algebra with an equivalent norm. In view of [2], Corollary 12 it suffices to prove that the set

$$\left\{\sum_{n=1}^{\infty} \frac{(i\tilde{h})^n}{n!} : \tilde{h} \in \tilde{A},\ \tilde{h}^* = \tilde{h}\right\}$$

is bounded in $A^\sim$. First for any normal $x \in A$ we have by a C^*-norm property

$$\text{(6)} \qquad \frac{1}{\sqrt{2}}\|x\| \leqq \|x\|_2 \leqq 8\left(\left\|\frac{x+x^*}{2}\right\| + \left\|\frac{x-x^*}{2}\right\|\right) \leqq 16\|x^*x\|^{1/2} = 16\|x\|$$

which gives for every $h \in H(A)$

$$\text{(7)} \qquad \frac{1}{\sqrt{2}}\left\|\sum_{n=1}^{\infty}(ih)^n/n!\right\| \leqq \left\|\sum_{n=1}^{\infty}(ih)^n/n!\right\|_2 \leqq 16\left\|\sum_{n=1}^{\infty}(ih)^n/n!\right\| \leqq 32,$$

since for the normal $\sum_{n=1}^{\infty}(ih)^n/n! \in A^\sim$ the original norm can be extended by the previous equivalence and the quasi-unitary elements are of norms not greater than 2 in C^*-algebras. Let now a selfadjoint $h \in A$ and a positive number ε be given. Choose an $h \in H(A)$ which satisfies $\|h\|_2 \leqq \|\tilde{h}\|_2$ and $\|\tilde{h} - h\|_2 < \varepsilon \cdot e^{-\|\tilde{h}\|_2}$. Then (7) gives by a simple computation

$$\left\|\sum_{n=1}^{\infty}(i\tilde{h})^n/n!\right\|_2 \leqq \left\|\sum_{n=1}^{\infty}(ih)^n/n!\right\|_2 + \varepsilon \leqq 32 + \varepsilon,$$

where

$$\left\|\sum_{n=1}^{\infty}\frac{1}{n!}[(i\tilde{h})^n - (ih)^n]\right\|_2 \leqq \sum_{n=1}^{\infty}\frac{1}{n!}\left\|\sum_{m=0}^{n-1}(i\tilde{h})^{n-m}(ih)^m - (i\tilde{h})^{n-m-1}(ih)^{m+1}\right\|_2 =$$

$$= \sum_{n=1}^{\infty}\frac{1}{n!}\left\|\sum_{m=0}^{n-1}(i\tilde{h})^{n-m-1}(i\tilde{h} - ih)(ih)^m\right\|_2 \leqq \|\tilde{h} - h\|_2 \sum_{n=1}^{\infty}\frac{\|\tilde{h}\|_2^{n-m-1}\|\tilde{h}\|_2^m}{(n-1)!} < \varepsilon$$

was used. This shows that 32 is an appropriate bound for the set considered abvoe. Thus A is a C^*-algebra with an equivalent norm, which agrees for every $x \in A$ with $\nu(x^*x)^{1/2}$ as well known from the C^*-condition. But thus (5) shows by the assumption for any $x \in A$

$$\nu(x^*x)^{1/2} = \|x^*x\|^{1/2} \leqq \|x\|. \tag{8}$$

We need show only the converse to (8) in the remainder. In case if A has an identity, for the C^*-norm we have by [3], (3. 7) Corollary the expression

$$\nu(x^*x)^{1/2} = \inf\left\{\sum_{j=1}^{n} |\lambda_j| : x = \sum_{j=1}^{n} \lambda_j \exp(i\tilde{h}_j),\ \tilde{h}_j = \tilde{h}_j^* \in \tilde{A};\ n = 1, 2, \ldots\right\}. \tag{9}$$

Assuming now $\sum_{j=1}^{n} |\lambda_j| < \nu(x^*x)^{1/2} + \varepsilon/2$ for some $\varepsilon > 0$ such that $x = \sum_{j=1}^{n} \lambda_j \exp(i\tilde{h}_j)$ holds with $\tilde{h}_j \in A^{\sim}$, $\tilde{h}_j^* = \tilde{h}_j$ $(j = 1, 2, \ldots, n)$, we can choose normal $x_j \in A$ which satisfy $\|x_j\| = 1$, $\|\exp(i\tilde{h}_j) - x_j\|_2 < \varepsilon/2\sqrt{2} \sum_{j=1}^{n} |\lambda_j|$ for $j = 1, 2, \ldots, n$. Then using (6) we have

$$\|x\| \leqq \left\|x - \sum_{j=1}^{n} \lambda_j x_j\right\| + \left\|\sum_{j=1}^{n} \lambda_j x_j\right\| < \sqrt{2} \sum_{j=1}^{n} |\lambda_j| \, \|\exp(i\tilde{h}_j) - x_j\|_2 + \sum_{j=1}^{n} |\lambda_j| <$$
$$< \nu(x^*x)^{1/2} + \varepsilon.$$

Since ε was an arbitrary positive number, the converse to (8) is valid as claimed. Suppose finally that A has not an identity. Then analogously

$$\nu(x^*x)^{1/2} = \inf\left\{\sum_{j=1}^{n} |\lambda_j| : x = \sum_{j=1}^{n} \lambda_j \sum_{m=1}^{\infty} (i\tilde{h}_j)^m/m! :\ \tilde{h}_j = \tilde{h}_j^* \in \tilde{A},\ (j = 1, 2, \ldots, n),\ n = 1, 2, \ldots\right\}$$

holds where $\sum_{j=1}^{n} \lambda_j = 0$ is automatically satisfied. The proof of the converse to (8) can be done in an analogous way. The proof of the theorem is complete.

References

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