On an identity of Shih-Chieh Chu

By LAJOS TAKÁCS in Cleveland (Ohio, U.S.A.)*)

Dedicated to Professor Béla Szőkefalvi-Nagy on the occasion of his sixtieth birthday July 29, 1973

1. On a formula of Shih-Chieh Chu. In a Chinese treatise of 1303 SHIH-CHIEH CHU [7] described the arithmetic triangle as an ancient invention for determining the terms in the expansion of $(a+b)^n$ where $n=1, 2, \ldots$. In his treatise Shih-Chieh Chu obtained several remarkable relations for binomial coefficients, but gave no proofs. One of the identities, attributed to Shih-Chieh Chu, can be expressed in modern notation as follows: If k and n are nonnegative integers, then

$$\sum_{j=0}^{k} \binom{k}{j}^2 \binom{n+2k-j}{2k} = \binom{n+k}{k}^2. \tag{1}$$

As usual, the binomial coefficient $\binom{x}{j}$ is defined for every real or complex x as follows:

$$\binom{x}{j} = \frac{x(x-1)\ldots(x-j+1)}{j!} \qquad (j=1,2,\ldots),$$

$$\binom{x}{0} = 1, \quad \binom{x}{j} = 0 \qquad (j=-1,-2,\ldots).$$

Since both sides of (1) are polynomials of degree $2k$ in the variable n, it follows that (1) also holds if n is any real or complex number.

More details about the rich contents of the treatise of SHIH-CHIEH CHU [7] can be found in the books by Y. MIKAMI [22 pp. 89—98, 124] and J. NEEDHAM and L. WANG [24 pp. 41, 46—47, 133—141]. In 1867 JEN-SHOU LI [18] brought the identity (1) to light in the fourth part of his book entitled Mathematical Studies. The problem of how this identity was found and how it can be proved aroused the interest of several mathematicians.

*) This research was supported by a grant from the National Science Foundation.

In 1937 Yung Chang, a scholar of the history of mathematics, mentioned the identity (1) to G. Szekeres who called the attention of P. Turán to it. Both G. Szekeres and P. Turán have found rather complicated proofs for (1). In 1939 their proofs were published in Chinese in an article by YUNG CHANG [6]. In 1954 P. TURÁN [35] published a Hungarian version of his proof.

In his proof G. Szekeres used mathematical induction and some known summation formulas for the binomial coefficients and proved (1) as a particular case of a more general relation. P. Turán's proof is analytical and based on some properties of the Legendre polynomials. By using Hurwitz's formula and Rodrigues's formula for the Legendre polynomials, P. Turán demonstrated that

$$(2) \qquad \sum_{j=0}^{k} \binom{k}{j}\binom{k+j}{j}(x-1)^{k-j} = \sum_{j=0}^{k} \binom{k}{j}^2 x^j$$

holds for every x and $k=0, 1, 2, \dots$, and by calculating the derivative

$$\frac{1}{k!}\frac{d^k}{dz^k}\left(\frac{z^k}{(x+z)^{k+1}}\right)_{z=-1}$$

in two different ways he proved that

$$(3) \qquad \sum_{n=0}^{\infty} \binom{n+k}{k}^2 x^n = (1-x)^{-2k-1} \sum_{j=0}^{k} \binom{k}{j}^2 x^j$$

holds for $|x|<1$. If we form the coefficient of x^n in the power series expansion of the right-hand side of (3), then we obtain (1).

It is interesting to mention that if we form the coefficient of z^k in both forms of the polynomial

$$(1+z)^k[1+z+(x-1)z]^k = (1+z)^k(1+xz)^k,$$

then we get (2). Formula (2) is a particular case of a more general result found in 1947 by W. LJUNGGREN [20]. Furthermore, formula (3) is a particular case of Saalschütz's theorem for hypergeometric functions. The general case of (3) was found in 1890 by L. SAALSCHÜTZ [28], [29]. (See also W. N. BAILEY [1 p. 9] and A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER and F. G. TRICOMI [8 pp. 65—66].) Let us mention that formula (3) was used in 1943 by O. BOTTEMA and S. C. VAN VEEN [4] in their studies of the game of billiard. (See also Problem 10 in Chapter 11 of W. FELLER [9].)

Subsequent to the investigations of G. Szekeres and P. Turán, simple proofs were found for (1) by L. TAKÁCS [34], J. SURÁNYI [31], [32], G. HUSZÁR [12], J. MÁTÉ [21], L. CARLITZ [5], J. SEITZ (see [16]), J. KAUCKÝ [14], [15], [16], [17], T. S. NANJUNDIAH [23] and R. L. GRAHAM and J. RIORDAN [11]. Most of these proofs are elementary and based on some well-known summation formulas for binomial coefficients. L. CARLITZ [5] showed that (1) immediately follows from a summation formula found in 1890 by L. SAALSCHÜTZ [28]. J. SEITZ and J. KAUCKÝ (see [16]) demonstrated that (3) can also be obtained by calculating

$$\frac{1}{k!}\frac{\partial^{2k}}{\partial x^k \partial z^k}\left(\frac{1}{1-xz}\right)_{z=1}$$

in two different ways.

By using a combinatorial approach, in 1955 J. SURÁNYI [31], [32] proved that if k, l and n are nonnegative integers, then we have

$$\sum_{j=0}^{k}\binom{k}{j}\binom{l}{j}\binom{n+k+l-j}{k+l}=\binom{n+k}{k}\binom{n+l}{l}. \tag{4}$$

If $l=k$, then (4) reduces to (1). Since both sides of (4) are polynomials of degree $k+l$ in the variable n, therefore (4) also holds if n is any real or complex number. This was demonstrated by LO-KENG HUA (see [31]) who proved that the left-hand side and the right-hand side of (4) have the same roots in n.

In 1958 T. S. NANJUNDIAH [23] proved a generalization of (1), which is equivalent to (4), and mentioned that by the same method it is possible to prove the following more general identity:

$$\sum_{j=0}^{k}\binom{l+x-y}{j}\binom{k+y-x}{k-j}\binom{y+j}{k+l}=\binom{x}{k}\binom{y}{l} \tag{5}$$

which holds for any nonnegative integers k and l and for any real or complex x and y. In fact T. S. Nanjundiah proved (5) only for $x=y$. In the case where x and y are nonnegative integers (5) was proved in 1970 by M. T. L. BIZLEY [2].

In this paper we shall prove that if k and m are nonnegative integers for which $0\leqq k\leqq m$, and l and c are real or complex numbers, then we have

$$\sum_{j=0}^{k}\binom{l}{j}\binom{m-l}{k-j}\binom{c+j}{m}=\binom{c}{m-k}\binom{c-m+k+l}{k}. \tag{6}$$

Since both sides of (6) are polynomials of degree k in l, therefore it is sufficient to prove (6) for $l=0, 1, \ldots, k$ and this implies that (6) holds for every l.

Formula (6) is a generalization of (1). If in (6) we replace (k, l, m, c) by $(k, k, 2k, n+k)$, and if we reverse the order in the sum, then we obtain (1). If in (6) we replace (k, l, m, c) by $(k, k, k+l, n+l)$, and if we reverse the order in the sum, then we obtain (4). If in (6) we replace (k, l, m, c) by $(k, l+x-y, k+l, y)$, then we obtain (5).

Now for any real or complex x and q let us define the Gaussian binomial coefficients $\begin{bmatrix} x \\ j \end{bmatrix}$ as follows:

$$\begin{bmatrix} x \\ j \end{bmatrix} = \begin{cases} \dfrac{(q^x-1)(q^{x-1}-1)\dots(q^{x-j+1}-1)}{(q^j-1)(q^{j-1}-1)\dots(q-1)} & \text{for} \quad q \neq 1, \\ \dfrac{x(x-1)\dots(x-j+1)}{j!} & \text{for} \quad q = 1, \end{cases}$$

if $j = 1, 2, \dots$; $\begin{bmatrix} x \\ 0 \end{bmatrix} = 1$, and $\begin{bmatrix} x \\ j \end{bmatrix} = 0$ if $j = -1, -2, \dots$.

By using the Gaussian binomial coefficients $\begin{bmatrix} x \\ j \end{bmatrix}$ with an arbitrary q we can generalize (6) in the following way

$$\sum_{j=0}^{k} \begin{bmatrix} l \\ j \end{bmatrix} \begin{bmatrix} m-l \\ k-j \end{bmatrix} \begin{bmatrix} c+j \\ m \end{bmatrix} q^{(k-j)(l-j)} = \begin{bmatrix} c \\ m-k \end{bmatrix} \begin{bmatrix} c-m+k+l \\ k \end{bmatrix} \tag{7}$$

which holds if k and m are nonnegative integers for which $0 \leqq k \leqq m$ and l and c are real or complex numbers. This follows from a result which was found in 1968 by E. M. Wright [36]. Actually, if we replace (k, l, m, c) by $(s, s-k, r+s-k, r+t-k)$ in (7), then we obtain Wright's formula (1) in his paper. For another proof of (7) we refer to H. W. Gould [10].

If we replace (k, l, m, c) by $(k, k, k+l, n+l)$ in (7), then we obtain a particular case of (7) which was found in 1965 by Gh. Pic [25].

We note that if in (6) and (7) we replace (k, l, m, c) by $(y, x+a-b, x+y, -a-1)$ and if we form the sum with respect to $i = a-j$, then we obtain the identities of R. P. Stanley [30] and H. W. Gould [10].

It is of some interest to mention that in 1965 Gh. Pic [25] generalized (1) by calculating

$$\frac{(z+x)^{\alpha+\beta}}{z^{\alpha}} \frac{d^k}{dz^k} \left(\frac{z^{k+\alpha}}{(z+x)^{k+\alpha+\beta+1}} \right)_{z=-1}$$

in two different ways. By using Rodrigues's formula for the Jacobi polynomials (see G. Szegő [33 pp. 58—99]) he obtained a generalization of (3). In the particular case where $\alpha = \beta = 0$ his result reduces to (3).

Finally, we note that by using the elements of the calculus of finite differences we can easily deduce a large number of formulas which have some resemblance to (6), but are actually of different types. Such a formula is the identity

$$\sum_{j=0}^{k} \binom{x}{j+r} \binom{j+r}{r} \binom{r+y}{k-j} = \binom{x}{r} \binom{x+y}{k}$$

which holds for every x and every y if k and r are nonnegative integers. We can easily prove this formula by using Newton's expansion. (See G. BOOLE [3 pp. 11—14] and CH. JORDAN [13 pp. 77—79].) More identities of similar types are given by many authors. See, for example, J. RIORDAN [26 pp. 14—15], [27 pp. 14—17], and D. J. LEWIS [19 p. 58].

2. A probabilistic proof for (6). We shall prove the identity (6) by using probabilistic methods. The following proof makes possible various generalizations of (6).

In the following proof we assume that k, l, m are nonnegative integers satisfying the inequalities $0 \leqq k \leqq m$ and $0 \leqq l \leqq m$ and that c is an arbitrary real or complex number.

In proving (6) we shall use the following result from the theory of probability. Suppose that a box contains l white balls and $m-l$ black balls. We draw k $(0 \leqq k \leqq m)$ balls without replacement. Let us suppose that every outcome of this random trial has the same probability, and denote by ν the number of white balls drawn. Then we have

$$\mathbf{P}\{\nu = j\} = \frac{\binom{k}{j}\binom{m-k}{l-j}}{\binom{m}{l}} = \frac{\binom{l}{j}\binom{m-l}{k-j}}{\binom{m}{k}} \tag{8}$$

for $j=0, 1, \ldots, k$, and

$$\mathbf{E}\left\{\binom{\nu}{r}\right\} = \sum_{j=r}^{k} \binom{j}{r} \mathbf{P}\{\nu = j\} = \frac{\binom{k}{r}\binom{l}{r}}{\binom{m}{r}} \tag{9}$$

for $r=0, 1, \ldots, k$.

Now we shall calculate $\binom{m}{k}\mathbf{E}\left\{\binom{c+\nu}{m}\right\}$ in two different ways. First, by (8) we obtain that

$$\binom{m}{k}\mathbf{E}\left\{\binom{c+\nu}{m}\right\} = \binom{m}{k}\sum_{j=0}^{k}\binom{c+j}{m}\mathbf{P}\{\nu = j\} = \sum_{j=0}^{k}\binom{c+j}{m}\binom{l}{j}\binom{m-l}{k-j}. \tag{10}$$

Second, by (9) we obtain that

$$\binom{m}{k}\mathbf{E}\left\{\binom{c+\nu}{m}\right\} = \binom{m}{k}\sum_{r=0}^{m}\binom{c}{m-r}\mathbf{E}\left\{\binom{\nu}{r}\right\} = \tag{11}$$

$$= \sum_{r=0}^{m}\binom{c}{m-k}\binom{c-m+k}{k-r}\binom{l}{r} = \binom{c}{m-k}\binom{c-m+k+l}{k}.$$

Here we used that

$$\binom{m}{k}\binom{c}{m-r}\binom{k}{r} = \binom{m}{r}\binom{c}{m-k}\binom{c-m+k}{k-r}.$$

If we compare (10) and (11), then we obtain (6) which was to be proved.

We note that if we calculate $\binom{m}{k}\mathbf{E}\left\{\binom{c-\nu}{m}\right\}$ in two different ways, then we obtain that

$$\sum_{j=0}^{k}\binom{l}{j}\binom{m-l}{k-j}\binom{c-j}{m} = \binom{c-k}{m-k}\binom{c-l}{k} \tag{12}$$

whenever k, l, m are nonnegative integers for which $0 \leqq k \leqq m$ and $0 \leqq l \leqq m$ and c is an arbitrary real or complex number. We can immediately see that (12) holds also if l is an arbitrary real or complex number.

In (12) the left-hand side can be obtained by (8). If we take into consideration that

$$\mathbf{E}\left\{\binom{c-\nu}{m}\right\} = (-1)^m \mathbf{E}\left\{\binom{m-c-1+\nu}{m}\right\},$$

then the right-hand side of (12) can be obtained by (11).

Of course (12) can also be obtained from (6) by substituting $m-c-1$ for c.

3. A generalization of (6). The previous probabilistic proof makes it possible to generalize (6) in various ways which we shall demonstrate by the following example.

Let us suppose that a box contains m cards each marked by one of the numbers $1, 2, \ldots, s$. Denote by l_i the number of cards marked i $(i=1, 2, \ldots, s)$. We have $l_1+l_2+\cdots+l_s = m$. We draw k $(0 \leqq k \leqq m)$ cards without replacement. Let us suppose that every outcome of this random trial has the same probability and denote by ν_i the number of cards marked i among the k cards drawn. Then we have

$$\mathbf{P}\{\nu_1 = j_1, \quad \nu_2 = j_2, \ldots, \nu_s = j_s\} = \frac{\binom{l_1}{j_1}\binom{l_2}{j_2}\cdots\binom{l_s}{j_s}}{\binom{m}{k}} \tag{13}$$

whenever $j_1+j_2+\cdots+j_s = k$, and

$$\mathbf{E}\left\{\binom{\nu_1}{r_1}\binom{\nu_2}{r_2}\cdots\binom{\nu_s}{r_s}\right\} = \frac{\binom{l_1}{r_1}\binom{l_2}{r_2}\cdots\binom{l_s}{r_s}\binom{m-r}{k-r}}{\binom{m}{k}} \tag{14}$$

where $r = r_1+r_2+\cdots+r_s$.

Let $c_1, c_2, \ldots, c_s$ be arbitrary real or complex numbers, and $1 \leqq t \leqq s$. By (13) we obtain that

$$\binom{m}{k} \mathbf{E}\left\{\binom{c_1+v_1}{m} \cdots \binom{c_t+v_t}{m}\right\} = \sum_{j_1+\cdots+j_s=k} \binom{c_1+j_1}{m} \cdots \binom{c_t+j_t}{m} \binom{l_1}{j_1} \cdots \binom{l_s}{j_s} =$$
$$\sum_{j_1+\cdots+j_t \leqq k} \binom{c_1+j_1}{m} \cdots \binom{c_t+j_t}{m} \binom{l_1}{j_1} \cdots \binom{l_t}{j_t} \binom{m-l_1-\cdots-l_t}{k-j_1-\cdots-j_t},$$

and by (14) we obtain that

$$\binom{m}{k} \mathbf{E}\left\{\binom{c_1+v_t}{m} \cdots \binom{c_t+v_t}{m}\right\} = \binom{m}{k} \sum_{r_1+\cdots+r_t \leqq k} \binom{c_1}{m-r_1} \cdots \binom{c_t}{m-r_t} \mathbf{E}\left\{\binom{v_1}{r_1} \cdots \binom{v_t}{r_t}\right\} =$$
$$= \sum_{r_1+\cdots+r_t \leqq k} \binom{c_1}{m-r_1} \cdots \binom{c_t}{m-r_t} \binom{l_1}{r_1} \cdots \binom{l_t}{r_t} \binom{m-r_1-\cdots-r_t}{k-r_1-\cdots-r_t}.$$

By comparing the above two formulas we obtain that if $k, l_1, \ldots, l_t$ are non-negative integers for which $k \leqq m$ and $l_1+\cdots+l_t \leqq m$, then

$$\sum_{j_1+\cdots+j_t \leqq k} \binom{l_1}{j_1} \cdots \binom{l_t}{j_t} \binom{m-l_1-\cdots-l_t}{k-j_1-\cdots-j_t} \binom{c_1+j_1}{m} \cdots \binom{c_t+j_t}{m} =$$
$$\sum_{r_1+\cdots+r_t \leqq k} \binom{l_1}{r_1} \cdots \binom{l_t}{r_t} \binom{m-r_1-\cdots-r_t}{k-r_1-\cdots-r_t} \binom{c_1}{m-r_1} \cdots \binom{c_t}{m-r_t} \tag{15}$$

holds for arbitrary real or complex $c_1, c_2, \ldots, c_t$. If $t=1$, then (15) can be reduced to (6).

Another generalization of (6) can be obtained in the following way. Let us suppose that we have s boxes and the ith box ($i=1, 2, \ldots, s$) contains l_i white balls and $m-l_i$ black balls. We draw k_1 balls from the first box, k_2 balls from the second box, and so on, k_s balls from the s-th box without replacement. Denote by v the total number of white balls drawn. If we calculate the expectation $\mathbf{E}\left\{\binom{c+v}{m}\right\}$ in two different ways, then we obtain another generalization of (6).

References

[1] W. N. Bailey, *Generalized Hypergeometric Series,* Cambridge University Press, 1935. (Reprinted by Stechert-Hafner, New York, 1964.)

[2] M. T. L. Bizley, A generalization of Nanjundiah's identity, *Amer. Math. Monthly,* **77** (1970), 863—865.

[3] G. Boole, *A Treatise on the Calculus of Finite Differences*. First edition, 1860. Second edition, 1872. (Second edition reprinted by Dover, New York, 1960.)

[4] O. Bottema and S. C. van Veen, Kansberekeningen bij het biljartspel, *Nieuw Archief voor Wiskunde*, **22** (1943), 15—33 and 123—158.

[5] L. Carlitz, On a problem in the history of Chinese mathematics, *Mat. Lapok*, **6** (1955), 219—220. (Hungarian)

[6] Yung Chang, To chi pi lei su chun (Proofs for summation formulas for binomial coefficients), *Science*, **23** (1939), 647—663. (Chinese)

[7] Shih-Chieh Chu, Ssu Yuan Yü Chien (Precious Mirror of the Four Elements), 1303. (Chinese)

[8] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*. Vol. I, McGraw Hill (New York, 1953).

[9] W. Feller, *An Introduction to Probability Theory and its Applications*. Vol. 1., John Wiley (New York, 1950). (Second edition 1957; Third edition 1968.)

[10] H. W. Gould, A new symmetrical combinatorial identity, *J. Combinatorial Theory*, (A) **13** (1972), 278—286.

[11] R. L. Graham and J. Riordan, The solution of a certain recurrence, *Amer. Math. Monthly*, **73** (1966), 604—608.

[12] G. Huszár, On a problem in the history of Chinese mathematics, *Matematikai Lapok*, **6** (1955), 36—38. (Hungarian)

[13] Ch. Jordan, *Calculus of Finite Differences* (Budapest, 1939). (Reprinted by Chelsea, New York, 1947.)

[14] J. Kaucký, Proof of the equation $\sum_{k=0}^{\nu}\binom{2N+\nu-k}{\nu-k}\binom{N}{k}^2=\binom{N+\nu}{\nu}^2$, *Sbornik Vojenské Tech. Akad. A. Zápotockého, Brno*, **5** (1957), 6—8. (Czech)

[15] J. Kaucký, Remarks on a paper of P. Turán, *Mat.-Fyz. Časopis Slovensk. Akad. Vied.*, **12**, (1962), 212—216. (Czech)

[16] J. Kaucký, On a problem in the history of Chinese mathematics, *Mat.-Fyz. Časopis Slovensk. Akad. Vied.*, **13** (1963), 32—40. (Czech)

[17] J. Kaucký, Une nouvelle démonstration élémentaire de la formule combinatoire de Li Jen-Shu, *Mat.-Fyz. Časopis Slovensk. Akad. Vied.*, **14** (1964), 50—53.

[18] Jen-Shou Li, Tsê Ku Hsi Chai Suan Hsüeh *(Tsê Ku Hsi Home Mathematical Studies)*, Nanking, 1867. (Chinese)

[19] D. J. Lewis, Diophantine equations: *p*-adic methods, *Studies in Number Theory*, Editor: W. J. LeVeque. *MAA Studies in Mathematics.*, Vol. 6. Mathematical Association of America, 1969, 25—75.

[20] W. Ljunggren, Et elementaert bevis for en formel av A. C. Dixon, *Norsk Mat. Tidskrift*, **29** (1947), 35—38.

[21] J. Máté, On a problem in the history of Chinese mathematics, *Mat. Lapok*, **7** (1956), 112—113. (Hungarian)

[22] Y. Mikami, *The Development of Mathematics in China and Japan*, Teubner (Leipzig, 1913). (Reprinted by Chelsea, New York, 1961.)

[23] T. S. Nanjundiah, Remark on a note of P. Turán, *Amer. Math. Monthly*, **65** (1958), 354.

[24] J. Needham and L. Wang, *Science and Civilisation in China*. Vol. 3., Mathematics and the Sciences of the Heavens and the Earth (Cambridge University Press, 1959).

[25] Gh. Pic, On a combinatorial formula, *Studia Univ. Babeş—Bolyai*, Ser. Math. Phys., **10**, No. 2, 7—15. (Romanian)

[26] J. Riordan, *An Introduction to Combinatorial Analysis*, John Wiley, New York, 1958.

[27] J. Riordan, *Combinatorial Identities*, John Wiley (New York, 1968).

[28] L. Saalschütz, Eine Summationsformel, *Z. für Math. und Phys.*, **35** (1890) 186—188.

[29] L. SAALSCHÜTZ, Über einen Specialfall der hypergeometrischen Reihe dritter Ordnung, *Z. für Math. und Phys.*, **36** (1891), 278—295 and 321—327.

[30] R. P. STANLEY, *Ordered structures and partitions*, Ph. D. Dissertation (Harvard University, 1970).

[31] J. SURÁNYI, Remarks on a problem in the history of Chinese mathematics, *Mat. Lapok*, **6** (1955), 30—35. (Hungarian)

[32] J. SURÁNYI, On a problem of old Chinese mathematics, *Publ. Math. Debrecen*, **4** (1956), 195—197.

[33] G. SZEGŐ, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, Vol. XXIII, American Mathematical Society (Providence, R. I., 1939). (Second ed. 1958. Third ed. 1967.)

[34] L. TAKÁCS, Remark on a paper of P. Turán entitled "On a problem in the history of Chinese mathematics", *Mat. Lapok*, **6** (1955), 27—29. (Hungarian)

[35] P. TURÁN, On a problem in the history of Chinese mathematics, *Mat. Lapok*, **5** (1954), 1—6. (Hungarian)

[36] E. M. WRIGHT, An identity and applications, *Amer. Math. Monthly*, **75** (1968), 711—714.

(Received November 29, 1972)