On logarithmic concave measures and functions

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Dedicated to Béla Szőkefalvi-Nagy on his 60th birthday

1. Introduction. The purpose of the present paper is to give a new proof for the main theorem proved in [3] and develop further properties of logarithmic concave measures and functions. Having in mind the applications of our theory to mathematical programming, we restrict ourselves to functions and measures in finite dimensional Euclidean spaces.

A function f defined on R^n is said to be logarithmic concave if for every pair of vectors $\mathbf{x}_1, \mathbf{x}_2 \in R^n$ and for every $0<\lambda<1$ we have

$$f(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \geqq (f(\mathbf{x}_1))^{\lambda}(f(\mathbf{x}_2))^{1-\lambda}. \tag{1.1}$$

A measure defined on the measurable subsets of R^n is logarithmic concave if for every pair A, B of convex subsets of R^n and for every $0<\lambda<1$, we have the following inequality

$$P(\lambda A + (1-\lambda)B) \geqq (P(A))^{\lambda}(P(B))^{1-\lambda}, \tag{1.2}$$

where the sign + means Minkowski addition of sets.

If the function f is logarithmic concave in R^n and $f \not\equiv 0$ then it can be written as $f(\mathbf{x}) = e^{-Q(\mathbf{x})}$ $(\mathbf{x} \in R^n)$ where $Q(\mathbf{x})$ is convex in the entire space and the value $+\infty$ is also allowed for the function Q. The set where f is positive, is convex and f is clearly continuous in the interior of this set.

The above-mentioned main theorem is repeated below in its original form.

Theorem 1. *Let Q be a convex function defined on the entire n-dimensional space. Suppose that $Q(\mathbf{x}) \geqq a$ where a is some real number. Let $\psi(z)$ be a function defined on the infinite interval $[a, \infty)$. Suppose that $\psi(z)$ is nonnegative, nonincreasing, differentiable and $-\psi'(z)$ is logarithmic concave. Consider the function $f(\mathbf{x}) = \psi(Q(\mathbf{x}))$ $(\mathbf{x} \in R^n)$ and suppose that it is a probability density i.e.*

$$\int_{R_n} f(\mathbf{x})\, d\mathbf{x} = 1. \tag{1.3}$$

Denote by $P(C)$ the integral of $f(\mathbf{x})$ over the measurable subset C of R^n. If A and B are any two convex sets in R^n and $0<\lambda<1$, then the inequality (1. 2) *holds.*

This theorem remains true without the assumption (1. 3). In fact the theorem is obviously true if the integral on the left hand side in (1. 3) is an arbitrary non-negative number. If this integral equals infinity then first we apply the theorem for the following function

$$f_T(\mathbf{x}) = f(\mathbf{x}) \text{ if } \|\mathbf{x}\| < T \text{ and } f_T(\mathbf{x}) = 0 \text{ otherwise},$$

where T is a positive number. The integral of f_T is finite over the space R^n hence we have (1. 2) with the measure P_T generated by f_T. Since

$$P(C) = \lim_{T\to\infty} P_T(C)$$

for every measurable set $C\subset R^n$, the inequality (1. 2) is satisfied with the measure generated by f too.

A second remark concerning Theorem 1 is the following: any function $\psi(z)$ satisfying the requirements of the theorem is itself logarithmic concave. In fact the finiteness of the integral of the function $\psi(Q(\mathbf{x}))$ over the space R^n implies that $\lim_{z\to\infty} \psi(z)=0$, hence

$$\psi(z) = \int_z^\infty [-\psi'(x)]\,dx \qquad (z \geqq a). \tag{1.4}$$

Consider the measure defined on the measurable subsets of R^1 generated by the logarithmic concave function

$$g(x) = -\psi'(x) \quad \text{if} \quad x\geqq a \quad \text{and} \quad g(x)=0 \quad \text{otherwise}.$$

The logarithmic concavity of this function implies that (see Theorem 3 in [3]) for any interval A of R^1, the following function of the variable z

$$\int_{A+z} g(x)\,dx \quad (-\infty<z<\infty) \tag{1.5}$$

is logarithmic concave. Since the functions (1.4) and (1. 5) coincide for $z\geqq a$, if $A=[0, \infty)$, our statement is proved. The function $\psi(z)$ can be written as

$$\psi(z) = e^{-s(z)}\,(z \geqq a),$$

where $s(z)$ is convex and nondecreasing. Any convex and nondecreasing function of a convex function is also convex hence $s(Q(\mathbf{x}))= S(\mathbf{x})$ is a convex function in R^n and

$$f(\mathbf{x}) = e^{-S(\mathbf{x})}\,(\mathbf{x}\in R^n).$$

In view of these two remarks, Theorem 1 can be reformulated in the following form, including the case of unbounded measures.

Theorem 2. *If the measure P, defined on the measurable subsets of R^n, is generated by a logarithmic concave function, then the measure P is also logarithmic concave.*

2. Integral inequalities. The proof of Theorem 1 published in [3] is based on the theorem of Brunn and on the following integral inequality proved also in [3]: if f, g are nonnegative and Borel measurable functions defined on R^1 and

$$r(t) = \sup_{x+y=2t} f(x)g(y) \qquad (-\infty < t < \infty)$$

then we have

$$\int_{-\infty}^{\infty} r(t)\,dt \geqq \Big(\int_{-\infty}^{\infty} f^2(x)\,dx\Big)^{\frac{1}{2}} \Big(\int_{-\infty}^{\infty} g^2(y)\,dy\Big)^{\frac{1}{2}}, \tag{2.1}$$

where the value $+\infty$ is also allowed for the integrals.

L. Leindler generalized this inequality in the following manner [2]. Let $f_1, \dots, f_k$ be nonnegative and Borel measurable functions defined on R^1 and define the function $r(t)$ $(t \in R^1)$ by the equality

$$r(t) = \sup_{\lambda_1 x_1 + \dots + \lambda_k x_k = t} f_1(x_1) \dots f_k(x_k),$$

where $\lambda_1, \dots, \lambda_k$ are positive constants satisfying the equality $\lambda_1 + \dots + \lambda_k = 1$. Then the function $r(t)$ $(t \in R^1)$ is also Borel measurable and we have the following inequality:

$$\int_{\infty-}^{\infty} r(t)\,dt \geqq \Big(\int_{-\infty}^{\infty} f_1^{\frac{1}{\lambda_1}}(x_1)\,dx_1\Big)^{\lambda_1} \dots \Big(\int_{-\infty}^{\infty} f_k^{\frac{1}{\lambda_k}}(x_k)\,dx_k\Big)^{\lambda_k}. \tag{2.2}$$

Now we generalize the inequality (2. 2) for functions of n variables. This generalisation is formulated in the following

Theorem 3. *Let $f_1, \dots, f_k$ be nonnegative and Borel measurable functions defined on R^n and let*

$$r(\mathbf{t}) = \sup_{\lambda_1 \mathbf{x}_1 + \dots + \lambda_k \mathbf{x}_k = \mathbf{t}} f_1(\mathbf{x}_1) \dots f_k(\mathbf{x}_k) (\mathbf{t} \in R^n),$$

where $\lambda_1, \dots, \lambda_k$ are positive constants satisfying the equality $\lambda_1 + \dots + \lambda_k = 1$. Then the function $r(\mathbf{t})$ $(\mathbf{t} \in R^n)$ is also Borel measurable and we have the following inequality:

$$\int_{R^n} r(\mathbf{t})\,d\mathbf{t} \geqq \Big(\int_{R^n} f_1^{\frac{1}{\lambda_1}}(\mathbf{x}_1)\,d\mathbf{x}_1\Big)^{\lambda_1} \dots \Big(\int_{R^n} f_k^{\frac{1}{\lambda_k}}(\mathbf{x}_k)\,d\mathbf{x}_k\Big)^{\lambda_k}. \tag{2.3}$$

Proof. The proof of the measurability of the function $r(\mathbf{t})$ ($\mathbf{t} \in R^{(n)}$) goes in an entirely similar way as in the case of $n=1$, $k=2$ (see the proof of Theorem 1 in [3]).

We prove (2. 3) by induction. Suppose that it holds for $n-1$ and prove that it holds for n. Let $x_{1i}, x_{2i}, \dots, x_{ki}, t_i$ ($i=1, \dots, n$) denote the components of the vectors $\mathbf{x}_1, \dots, \mathbf{x}_k, \mathbf{t}$, respectively. Fixing the second, ..., nth components so that

$$t_i = \lambda_1 x_{1i} + \lambda_2 x_{2i} + \cdots + \lambda_k x_{ki} \qquad (i = 2, \dots, n), \tag{2. 4}$$

it follows that

$$r(t_1, t_2, \dots, t_n) \geqq \sup_{\lambda_1 x_{11} + \lambda_2 x_{21} + \cdots + \lambda_k x_{k1} = t_1} f_1(x_{11}, x_{12}, \dots, x_{1n}) \cdots f_k(x_{k1}, x_{k2}, \dots, x_{kn}).$$

By the application of the inequality (2. 2) it follows from here that

$$\int_{-\infty}^{\infty} r(t_1, t_2, \dots, t_n)\, dt_1 \geqq$$

$$\geqq \Big(\int_{-\infty}^{\infty} f_1^{\frac{1}{\lambda_1}}(x_{11}, x_{12}, \dots, x_{1n})\, dx_{11} \Big)^{\lambda_1} \cdots \Big(\int_{-\infty}^{\infty} f_k^{\frac{1}{\lambda_k}}(x_{k1}, x_{k2}, \dots, x_{kn})\, dx_{k1} \Big)^{\lambda_k}.$$

Taking into account (2. 4) we can write further that

$$\int_{-\infty}^{\infty} r(t_1, t_2, \dots, t_n)\, dt_1 \geqq \sup_{\substack{\lambda_1 x_{12} + \dots + \lambda_k x_{k2} = t_2 \\ \vdots \\ \lambda_1 x_{1n} + \dots + \lambda_k x_{kn} = t_n}} \Big(\int_{-\infty}^{\infty} f_1^{\frac{1}{\lambda_1}}(x_{11}, x_{12}, \dots, x_{1n})\, dx_{11} \Big)^{\lambda_1} \cdots$$
$$\cdots \Big(\int_{-\infty}^{\infty} f_k^{\frac{1}{\lambda_k}}(x_{k1}, x_{k2}, \dots, x_{kn})\, dx_{k1} \Big)^{\lambda_k}. \tag{2. 5}$$

Now we apply the inductive assumption that the inequality (2. 3) holds for functions of $n-1$ variables. This implies that the integral on the right hand side of (2. 5) with respect to $t_2, \dots, t_n$ is greater than or equal to the following product

$$\Big(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_1^{\frac{1}{\lambda_1}}(x_{11}, \dots, x_{1n})\, dx_{11} \dots dx_{1n} \Big)^{\lambda_1} \cdots$$
$$\cdots \Big(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_k^{\frac{1}{\lambda_k}}(x_{k1}, \dots, x_{kn})\, dx_{k1} \dots dx_{kn} \Big)^{\lambda_k}. \tag{2. 6}$$

Looking at (2. 5) we immediately see that the integral of $r(\mathbf{t})$ over the space R_n is greater than or equal to the product standing in (2. 6). Thus the theorem is proved

Remark. In what follows we need only that special case of the integral inequality (2. 3) where $k=2$ and the functions f_1, f_2 are logarithmic concave. The proof of this special case is very easy on the basis of the integral inequality (2. 1).

Below we give a sketch of this proof. We may restrict ourselves to the case of $n=1$ since Theorem 3 shows that the generalization for the case of $n>1$ is simple. Let $N=2^m$ where m is a positive integer and let $i+j=N$. By a subsequent application of (2. 1) we get

$$\int_{-\infty}^{\infty} \sup_{\frac{1}{N}(x_1+\dots+x_N)=t} f_1^{\frac{N}{i}}(x_1)\dots f_1^{\frac{N}{i}}(x_i) f_2^{\frac{N}{j}}(x_{i+1})\dots f_2^{\frac{N}{j}}(x_N)\,dt \geqq$$
$$\geqq \Big(\int_{-\infty}^{\infty} f_1^{\frac{N}{i}}(x)\,dx\Big)^{\frac{i}{N}} \Big(\int_{-\infty}^{\infty} f_2^{\frac{N}{j}}(x)\,dx\Big)^{\frac{j}{N}}.$$

The logarithmic concavity of f_1 and f_2 implies that the integrand on the left hand side is smaller than or equal to the following function

$$\sup_{\frac{i}{N}u+\frac{j}{N}v=t} f_1(u) f_2(v) \qquad (-\infty<t<\infty)$$

thus we have (2. 2) for $k=2$ and $\lambda=\frac{i}{N}$, $1-\lambda=\frac{j}{N}$. The assertion for arbitrary $\lambda\,(0<\lambda<1)$ follows from here by a continuity argument.

3. New proof and sharpening of Theorem 2. On the basis of Theorem 4 the proof of Theorem 2 is very simple. To do this let us define the functions f_1, f_2, f_3 as follows:

$$f_1(\mathbf{x}) = f(\mathbf{x}) \text{ if } \mathbf{x}\in A \text{ and } f_1(\mathbf{x})=0 \text{ otherwise,}$$
$$f_2(\mathbf{x}) = f(\mathbf{x}) \text{ if } \mathbf{x}\in B \text{ and } f_2(\mathbf{x})=0 \text{ otherwise,}$$
$$f_3(\mathbf{x}) = f(\mathbf{x}) \text{ if } \mathbf{x}\in \lambda A+(1-\lambda)B \text{ and } f_3(\mathbf{x})=0 \text{ otherwise.}$$

The logarithmic concavity of the function f implies that for every $\mathbf{t}$ we have

$$f_3(\mathbf{t}) \geqq \sup_{\lambda\mathbf{x}+(1-\lambda)\mathbf{y}=\mathbf{t}} (f_1(\mathbf{x}))^{\lambda}(f_2(\mathbf{y}))^{1-\lambda}.$$

Hence, applying Theorem 4, we obtain

$$\int_{\lambda A+(1-\lambda)B} f(\mathbf{t})\,d\mathbf{t} = \int_{R^n} f_3(\mathbf{t})\,d\mathbf{t} \geqq \Big(\int_{R^n} f_1(\mathbf{x})\,d\mathbf{x}\Big)^{\lambda}\Big(\int_{R^n} f_2(\mathbf{y})\,d\mathbf{y}\Big)^{1-\lambda} =$$
$$= \Big(\int_A f(\mathbf{x})\,d\mathbf{x}\Big)^{\lambda}\Big(\int_B f(\mathbf{y})\,d\mathbf{y}\Big)^{1-\lambda},$$

which is the required inequality.

Theorem 4. *Let P be a measure defined on the measurable subsets of R^n and generated by the logarithmic concave function f. Let A, B be two convex subsets of R^n with the property that $0<P(A)<\infty$, $0<P(B)<\infty$. Suppose that for every $\lambda\,(0<\lambda<1)$ the sets A and B can be decomposed as $A=A_1\cup A_2$, $B=B_1\cup B_2$, where $A_1\cap A_2=\emptyset$, $B_1\cap B_2=\emptyset$ so that the following conditions are satisfied.*

a) A_1, B_1 *are bounded closed convex sets*, A_2, B_2 *are convex sets.*

b) *The following relations hold*

$$[\lambda A_1+(1-\lambda)B_1]\cup[\lambda A_2+(1-\lambda)B_2]=\lambda A+(1-\lambda)B, \tag{3.1}$$

$$[\lambda A_1+(1-\lambda)B_1]\cap[\lambda A_2+(1-\lambda)B_2]=\emptyset, \tag{3.2}$$

$$[\lambda A_1+(1-\lambda)B_1]\cap A_1=\emptyset, \tag{3.3}$$

$$[\lambda A_1+(1-\lambda)B_1]\cap B_1=\emptyset. \tag{3.4}$$

c) *For the measures of the decomposing sets the following relations hold*

$$P(A_1)>0, \quad P(B_1)>0, \tag{3.5}$$

$$\frac{P(A_2)}{P(A_1)}=\frac{P(B_2)}{P(B_1)}. \tag{3.6}$$

d) f *is strictly logarithmic concave in the convex hull of* $A_1\cup B_1$ *i.e. the strict inequality holds in* (1. 1) *whenever* $\mathbf{x}_1, \mathbf{x}_2$ *are elements of the convex hull of* $A_1\cup B_1$ *and* $\mathbf{x}_1\neq\mathbf{x}_2$.

Under these conditions for every $\lambda\,(0<\lambda<1)$ *we have*

$$P(\lambda A+(1-\lambda)B)>(P(A))^{\lambda}(P(B))^{1-\lambda}.$$

Proof. Let λ be a number satisfying $0<\lambda<1$ and consider the subdivisions of the sets A, B belonging to this λ. Since A_1, B_1 are disjoint closed convex sets and f is strictly logarithmic concave in the convex hull of $A_1\cup B_1$, (3. 3) and (3. 4) imply that

$$f(\mathbf{t})>\sup_{\substack{\lambda\mathbf{x}+(1-\lambda)\mathbf{y}=\mathbf{t}\\ \mathbf{x}\in A_1,\,\mathbf{y}\in B_1}}(f(\mathbf{x}))^{\lambda}(f(\mathbf{y}))^{1-\lambda}. \tag{3.7}$$

Let $f_1(\mathbf{x})=f(x)$ if $x\in A_1$ and $f_1(\mathbf{x})=0$ otherwise, $f_2(\mathbf{y})=f(\mathbf{y})$ if $\mathbf{y}\in B_1$ and $f_2(\mathbf{y})=0$ otherwise. Then for every $\mathbf{t}\in\lambda A_1+(1-\lambda)B_1$ we have by (3. 7):

$$f(\mathbf{t})>\sup_{\lambda\mathbf{x}+(1-\lambda)\mathbf{y}=\mathbf{t}}(f_1(\mathbf{x}))^{\lambda}(f_2(\mathbf{y}))^{1-\lambda}. \tag{3.8}$$

If $\mathbf{t}\notin\lambda A_1+(1-\lambda)B_1$ then the right hand side in (3. 8) equals 0. Hence it follows that

$$\begin{aligned} P(\lambda A_1+(1-\lambda)B_1) &= \int_{\lambda A_1+(1-\lambda)B_1} f(\mathbf{t})\,d\mathbf{t} > \\ &> \int_{\lambda A_1+(1-\lambda)B_1}\sup_{\lambda\mathbf{x}+(1-\lambda)\mathbf{y}=\mathbf{t}}(f_1(\mathbf{x}))^{\lambda}(f_2(\mathbf{y}))^{1-\lambda}\,d\mathbf{t} = \\ &= \int_{R^n}\sup_{\lambda\mathbf{x}+(1-\lambda)\mathbf{y}=\mathbf{t}}(f_1(\mathbf{x}))^{\lambda}(f_2(\mathbf{y}))^{1-\lambda}\,d\mathbf{t} \geqq \\ &\geqq \Big(\int_{R^n}f_1(\mathbf{x})\,d\mathbf{x}\Big)^{\lambda}\Big(\int_{R^n}f_2(\mathbf{y})\,d\mathbf{y}\Big)^{1-\lambda}=(P(A_1))^{\lambda}(P(B_1))^{1-\lambda}. \end{aligned} \tag{3.9}$$

Continuing the reasoning it follows from (3. 1) and (3. 2), (3. 9) and Theorem 2 that for every $\lambda(0<\lambda<1)$

$$P(\lambda A+(1-\lambda)B) = P(\lambda A_1+(1-\lambda)B_1)+P(\lambda A_2+(1-\lambda)B_2) >$$
$$> (P(A_1))^\lambda (P(B_1))^{1-\lambda}+(P(A_2))^\lambda (P(B_2))^{1-\lambda}.$$

Taking into account (3. 5) we can write

$$(P(A))^\lambda (P(B))^{1-\lambda} = (P(A_1))^\lambda (P(B_1))^{1-\lambda}+(P(A_2))^\lambda (P(B_2))^{1-\lambda}.$$

Thus the theorem is proved.

One of the most important application of Theorem 5 is expressed by the following

Theorem 5. *Let f be a logarithmic concave probability density in R^n. Denote by F the probability distribution function belonging to the density f. If f is positive and strictly logarithmic concave in an open convex set D then F is also strictly logarithmic concave in the set D.*

Proof. Let $\mathbf{u}$, $\mathbf{v}$ be elements of the interior of the set D and suppose that $\mathbf{u}\neq\mathbf{v}$. By the definition of the function F we can write that

$$\begin{aligned} F(\mathbf{u}) &= P(A), \quad \text{where} \quad A = \{\mathbf{x}|\mathbf{x} \leqq \mathbf{u}\}, \\ F(\mathbf{v}) &= P(B), \quad \text{where} \quad B = \{\mathbf{x}|\mathbf{x} \leqq \mathbf{v}\}. \end{aligned} \tag{3.10}$$

Given a $\lambda(0<\lambda<1)$, the following equality is obviously true

$$F(\lambda\mathbf{u}+(1-\lambda)\mathbf{v}) = P(\lambda A+(1-\lambda)B). \tag{3.11}$$

Let us define the sets A_1, B_1, A_2, B_2 in the following way

$$A_1 = \left\{\mathbf{x}|\mathbf{x} \leqq \mathbf{u},\ \sum_{i=1}^n x_i \geqq \sum_{i=1}^n u_i-\varepsilon\right\}, \qquad B_1 = \left\{\mathbf{x}|\mathbf{x} \leqq \mathbf{v},\ \sum_{i=1}^n x_i \geqq \sum_{i=1}^n v_i-\delta\right\},$$
$$A_2 = A-A_1, \quad B_2 = B-B_1,$$

where ε and δ are fixed positive numbers. Obviously $P(A_1)>0$, $P(B_1)>0$.

Conditions (3. 1) and (3. 2) are satisfied for every positive ε, δ while conditions (3. 3), (3. 4) are satisfied for sufficiently small positive numbers ε, δ. This statement follows from the following equalities:

$$\lambda A_1+(1-\lambda)B_1 = \left\{\mathbf{x}|\mathbf{x} \leqq \lambda\mathbf{u}+(1-\lambda)\mathbf{v},\ \sum_{i=1}^n x_i \geqq \sum_{i=1}^n (\lambda u_i+(1-\lambda)v_i)-\lambda\varepsilon-(1-\lambda)\delta\right\},$$

$$\lambda A_2+(1-\lambda)B_2 = \left\{\mathbf{x}|\mathbf{x} \leqq \lambda\mathbf{u}+(1-\lambda)\mathbf{v},\ \sum_{i=1}^n x_i < \sum_{i=1}^n (\lambda u_i+(1-\lambda)v_i)-\lambda\varepsilon-(1-\lambda)\delta\right\}.$$

Let us fix an ε_0 and a δ_0 having this property. Then if (3. 6) holds true, we are ready.

If on the other hand (3. 6) is not satisfied, then since $P(A_1)$ is continuous in ε, $P(B_1)$ is continuous in δ and

$$\lim_{\varepsilon\to 0} P(A_1) = \lim_{\delta\to 0} P(B_1) = 0,$$

we can find positive numbers ε_1, δ_1 such that $\varepsilon_1 \leqq \varepsilon_0$, $\delta_1 \leqq \delta_0$ and (3. 6) is satisfied with these. Thus in view of (3. 10) and (3. 11) our theorem follows from Theorem 4.

4. Further properties of logarithmic concave functions. In this section we mention three theorems concerning logarithmic concave functions. The proofs are based on the integral inequality (2. 3).

Theorem 6. *Let $f(\mathbf{x}, \mathbf{y})$ be a function of $n+m$ variables where $\mathbf{x}$ is an n-component and $\mathbf{y}$ is an m-component vector. Suppose that f is logarithmic concave in R^{n+m} and et A be a convex subset of R^m. Then the function of the variable $\mathbf{x}$:*

$$\int_A f(\mathbf{x}, \mathbf{y})\, d\mathbf{y}$$

lis logarithmic in the entire space R^n.

Proof. Let $\mathbf{x}_1, \mathbf{x}_2 \in R^n$ and $0<\lambda<1$. Define the functions $f_1(\mathbf{y})$, $f_2(\mathbf{y})$, $f_3(\mathbf{y})$ as follows:

$$f_1(\mathbf{y}) = f(\mathbf{x}_1, \mathbf{y}) \text{ if } \mathbf{y}\in A, \text{ and } f_1(\mathbf{y}) = 0 \text{ otherwise,}$$

$$f_2(\mathbf{y}) = f(\mathbf{x}_2, \mathbf{y}) \text{ if } \mathbf{y}\in A, \text{ and } f_2(\mathbf{y}) = 0 \text{ otherwise,}$$

$$f_3(\mathbf{y}) = f\big(\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2, \mathbf{y}\big) \text{ if } \mathbf{y}\in A, \text{ and } f_3(\mathbf{y}) = 0 \text{ otherwise.}$$

Since f is logarithmic concave in R^{n+m} and A is a convex set in R^m, we have

$$f_3(\mathbf{y}) \geqq \sup_{\lambda\mathbf{u}+(1-\lambda)\mathbf{v}=\mathbf{y}} (f_1(\mathbf{u}))^\lambda (f_2(\mathbf{v}))^{1-\lambda}.$$

Hence by Theorem 3 it follows that

$$\int_A f(\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2, \mathbf{y})\, d\mathbf{y} = \int_{R^m} f_3(\mathbf{y})\, d\mathbf{y} \geqq$$

$$\geqq \Big(\int_{R^m} f_1(\mathbf{u})\, d\mathbf{u}\Big)^\lambda \Big(\int_{R^m} f_2(\mathbf{v})\, d\mathbf{v}\Big)^{1-\lambda} = \Big(\int_A f(\mathbf{x}_1, \mathbf{y})\, d\mathbf{y}\Big)^\lambda \Big(\int_A f(\mathbf{x}_2, \mathbf{y})\, d\mathbf{y}\Big)^{1-\lambda}.$$

Thus the theorem is proved.

Theorem 7. *Let f, g be logarithmic concave functions defined in the space R^n. Then the convolution of these functions i.e.*

$$\int_{R^n} f(\mathbf{x}-\mathbf{y})\, g(\mathbf{y})\, d\mathbf{y}$$

is also logarithmic concave in the entire space R^n.

Proof. The theorem is a consequence of Theorem 7. In fact the function $f(\mathbf{x}-\mathbf{y})g(\mathbf{y})$ is a logarithmic concave function of the $2n$ variables contained in the vectors $\mathbf{x}$ and $\mathbf{y}$. Applying Theorem 6 for this function and for the convex set $A=R^n$, we obtain the assertion of Theorem 7.

For the case $n=1$ the assertion was proved by IBRAGIMOV [1] in 1956. The following theorem is an immediate consequence of Theorem 6. It is mentioned separately for completeness.

Theorem 8. *If f is a logarithmic concave multivariate probability density, then all marginal densities are also logarithmic concave.*

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