On quasi-equivalence and quasi-similarity

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Dedicated to Professor Béla Szőkefalvi-Nagy on his 60th birthday

Introduction

The object of this note is to show how the theory of Jordan models for C_0 contractions of finite defect, which was developed by Sz.-Nagy and Foiaş [10—17], may be approached from the relation of quasi-equivalence for H^∞ matrices [6]. We thereby give a different approach to the Jordan model theory and in so doing establish a conjecture of Sz.-Nagy and Foiaş [15] on the relation between the inner functions in the Jordan model of an operator T and its characteristic operator function Θ_T. Our results show that the analogy with the finite dimensional situation is complete.

1. Preliminaries

Let $\mathfrak{E}$ be a separable complex Hilbert space and m normalized Lebesgue measure on the unit circle C of the complex plane. Then $L^2(\mathfrak{E})$ is the Hilbert space of all weakly measurable functions from C to $\mathfrak{E}$ having square integrable norm, and $H^2(\mathfrak{E})$ is the corresponding Hardy subspace. If χ is the identity function on C, then the bilateral shift operator U on $L^2(\mathfrak{E})$ is given by

$$Uf=\chi f \quad (f\in L^2(\mathfrak{E})),$$

where the operation is that of pointwise multiplication. The unilateral shift U_+ on $H^2(\mathfrak{E})$ is simply the restriction of U to $H^2(\mathfrak{E})$. The above is discussed in detail by Helson [4] and Sz.-Nagy and Foiaş [18].

The algebra of weakly measurable, essentially bounded functions from C to the algebra $\mathscr{B}(\mathfrak{E})$ of bounded operators on $\mathfrak{E}$ is $L^\infty(\mathscr{B}(\mathfrak{E}))$. A function Θ in $L^\infty(\mathscr{B}(\mathfrak{E}))$ is said to be analytic if

$$\int (\Theta(z)f, g)z^n\,dm(z)=0 \quad (f,g\in\mathfrak{E},\ n=1,2,\dots),$$

and $H^\infty(\mathscr{B}(\mathfrak{E}))$ is the algebra of analytic functions in $L^\infty(\mathscr{B}(\mathfrak{E}))$.

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If Θ is in $H^\infty(\mathscr{B}(\mathfrak{E}))$, then there are two operators that are naturally associated with it. Each sends a function u to Θu, where

$$\Theta u(z) = \Theta(z)u(z) \quad (z \in C),$$

but the domain of one is $L^2(\mathfrak{E})$ whereas that of the other is $H^2(\mathfrak{E})$. The former is the associated analytic Laurent operator and the latter the associated analytic Toeplitz operator. We will be somewhat imprecise and use the same notation for each of these operators as for the function that induces it, relying on the context to make it clear which is intended. If the operator Θ on $H^2(\mathfrak{E})$ is a partial isometry, then the analytic function inducing it will be called inner. This usage does not quite conform to that of Sz.-Nagy and Foiaş [18, p. 190], who use the term inner for an analytic function whose values are operators from one Hilbert space $\mathfrak{E}_1$ to another $\mathfrak{E}_2$ such that the induced analytic Toeplitz operator from $H^2(\mathfrak{E}_1)$ to $H^2(\mathfrak{E}_2)$ is isometric. The difference is inessential in that with either definition the typical invariant subspace of U_+ is the range of an analytic Toeplitz operator induced by some inner function ([1], [3], [5], [7] and [8]).

For each inner Θ let $S(\Theta)$ be the compression of U_+ to the orthogonal complement $\mathfrak{H}(\Theta)$ of $\Theta H^2(\mathfrak{E})$. It is now well known that every contraction T on a separable Hilbert space such that $\{T^{*n}\}$ converges to zero strongly is unitarily equivalent to some $S(\Theta)$ [18].

Our present interest centers on the case where $\mathfrak{E}$ is finite dimensional and Θ is an isometric operator on $H^2(\mathfrak{E})$. If Θ is purely contractive [18, p. 188], and if $\mathfrak{E}$ has dimension N, then both defect indices of $S(\Theta)$ are N, and $S(\Theta)$ belongs to class $C_0(N)$. In fact, up to unitary equivalence the most general contraction of class $C_0(N)$ arises this way [11]. In general, we will not require Θ to be purely contractive, and thus the resulting $S(\Theta)$ is of class $C_0(M)$ for some positive integer $M \leqq N$.

An operator X from a Hilbert space $\mathfrak{H}_1$ to another $\mathfrak{H}_2$ is called a quasi-affinity in case it is one to one and has dense range. An operator T_1 on $\mathfrak{H}_1$ is a quasi-affine transform of an operator T_2 on $\mathfrak{H}_2$ in case there exists a quasi-affinity X from $\mathfrak{H}_1$ to $\mathfrak{H}_2$ such that

$$XT_1 = T_2 X,$$

in which case we write $T_1 \prec T_2$. If $T_1 \prec T_2$ and $T_2 \prec T_1$, then T_1 and T_2 are called quasi-similar. Again, for a more detailed discussion of these ideas refer to the text by Sz.-Nagy and Foiaş [18].

Two functions Θ_1 and Θ_2 in $H^\infty(\mathscr{B}(\mathfrak{E}))$ are said to be *equivalent* in case there exist two invertible functions Δ and Λ in $H^\infty(\mathscr{B}(\mathfrak{E}))$ such that

$$\Delta\Theta_1 = \Theta_2 \Lambda.$$

A function Δ in $H^\infty(\mathscr{B}(\mathfrak{E}))$ is said to have a scalar H^∞ function δ as a scalar multiple if there exists a function Ω in $H^\infty(\mathscr{B}(\mathfrak{E}))$ such that

$$\Omega\Delta=\Delta\Omega=\delta I_{\mathfrak{E}},$$

where $I_{\mathfrak{E}}$ is the identity operator on $\mathfrak{E}$. Finally, the following concepts introduced in [6] are fundamental to our present considerations. A *quasi-unit* $Д$ is a subset of $H^\infty(\mathscr{B}(\mathfrak{E}))$ with the property that the collection of all scalar multiples of the functions in $Д$ is nonempty and relatively prime, i.e. has no nonconstant common inner factor. If Θ_1 and Θ_2 belong to $H^\infty(\mathscr{B}(\mathfrak{E}))$ and if there exist quasi-units $Д$ and $Л$ such that

$$Д\Theta_1=\Theta_2 Л,$$

then Θ_1 and Θ_2 are called *quasi-equivalent*.

In this paper we will study the relation between quasi-equivalence and quasi-similarity, but we pause to note that in one direction at least the relation between equivalence and similarity parallels the finite dimensional situation.

Theorem 1. *Let Θ_1 and Θ_2 be inner functions in $H^\infty(\mathscr{B}(\mathfrak{E}))$. If Θ_1 and Θ_2 are equivalent, then $S(\Theta_1)$ and $S(\Theta_2)$ are similar.*

Proof. Let Δ and Λ be invertible functions in $H^\infty(\mathscr{B}(\mathfrak{E}))$ such that

$$\Delta\Theta_1=\Theta_2\Lambda. \tag{1}$$

It follows from (1) that $\Delta\Theta_1 H^2(\mathfrak{E})\subset\Theta_2 H^2(\mathfrak{E})$, and hence

$$P_2\Delta P_1=P_2\Delta, \tag{2}$$

where P_j is the orthogonal projection of $H^2(\mathfrak{E})$ onto $\mathfrak{H}(\Theta_j)$ for $j=1, 2$. Define an operator X by

$$X=P_2\Delta|\mathfrak{H}(\Theta_1). \tag{3}$$

For every $f\in\mathfrak{H}(\Theta_1)$ we have by (2) and (3) that

$$XS(\Theta_1)f=P_2\Delta P_1 U_+f=P_2\Delta U_+f=P_2U_+\Delta f=P_2U_+P_2\Delta f=S(\Theta_2)Xf,$$

where the next to last equality follows from the invariance of $\Theta_2 H^2(\mathfrak{E})$ under U_+. It remains only to show that X is invertible. We have by hypothesis that Δ is invertible; therefore, if v is an element of $\mathfrak{H}(\Theta_2)$, then there exists a unique element u in $H^2(\mathfrak{E})$ such that

$$v=\Delta u=P_2\Delta u.$$

Employing (2), we have

$$v=P_2\Delta P_1 u=XP_1 u;$$

thus X is onto. If $Xu=0$ for some u in $\mathfrak{H}(\Theta_1)$, then $\Delta u=\Theta_2 v$ for some v in $H^2(\mathfrak{E})$, and since Λ is invertible there exists some v' in $H^2(\mathfrak{E})$ such that $\Lambda v'=v$. Hence by (1),

$$\Delta(u-\Theta_1 v') = \Delta u - \Theta_2 \Lambda v' = 0.$$

But Δ is invertible and u is orthogonal to $\Theta_1 v'$; therefore, $u=0$, and X is one-to-one. Thus X is invertible.

It would be interesting to know to what extent Theorem 1 has a converse; it is known [2] that defect indices are not similarity invariants.

2. Main theorems

The following result is an analog of Theorem 1 for quasi-equivalence.

Theorem 2. *Let $\mathfrak{E}$ be finite dimensional and suppose that Θ_1 and Θ_2 are inner functions in $H^\infty(\mathscr{B}(\mathfrak{E}))$ that induce isometries on $H^2(\mathfrak{E})$. If Θ_1 and Θ_2 are quasi-equivalent, then $S(\Theta_1)$ and $S(\Theta_2)$ are quasi-similar.*

Proof. Since quasi-equivalence is an equivalence relation [6, Cor. 3. 2], it will suffice to prove $S(\Theta_1)\prec S(\Theta_2)$. The facts that E is finite dimensional and that Θ_1 and Θ_2 induce isometries on $H^2(E)$ imply Θ_1 and Θ_2 are each unitary valued a.e. and consequently their determinants are nonzero a.e. By Corollary 3. 3 of [6], there exist functions Δ and Λ in $H^\infty(\mathscr{B}(\mathfrak{E}))$ whose determinants are relatively prime to those of Θ_2 and Θ_1, and such that

$$\Delta\Theta_1=\Theta_2\Lambda.$$

Define X as in (3). The same reasoning used previously implies

$$XS(\Theta_1)=S(\Theta_2)X;$$

therefore, we need only show that X is a quasi-affinity.

Suppose v in $\mathfrak{H}(\Theta_2)$ is orthogonal to the range of X, and let u be any vector in $H^2(\mathfrak{E})$. There exist u' in $\mathfrak{H}(\Theta_1)$ and u'' in $H^2(\mathfrak{E})$ such that

$$u = u'+\Theta_1 u''.$$

By supposition,

$$(v, \Delta u')=(P_2 v\ \Delta u')=(v, P_2\,\Delta u')=v, Xu')=0,$$

and since $\Delta\Theta_1=\Theta_2\Lambda$, we have

$$(v, \Delta\Theta_1 u'')=(v, \Theta_2 \Lambda u'')=0.$$

Thus v is orthogonal to $\Delta H^2(\mathfrak{E})$, which includes $(\det \Delta)H^2(\mathfrak{E})$. But v is also orthogonal to $\Theta_2 H^2(\mathfrak{E})$, which includes $(\det \Theta_2)H^2(\mathfrak{E})$. Since $\det \Delta$ and $\det \Theta_2$ are relatively prime, it follows that $\Delta H^2(\mathfrak{E})$ and $\Theta_2 H^2(\mathfrak{E})$ span $H^2(\mathfrak{E})$ (see [1]); thus $v=0$. Consequently X has dense range.

Now consider an arbitrary $u\in\mathfrak{H}(\Theta_1)$ such that $Xu=0$; i.e. for which $\Delta u\in \Theta_2 H^2(\mathfrak{E})$. As mentioned earlier, each of Θ_1 and Θ_2 is unitary valued a.e., and therefore, the operators Θ_1 and Θ_2 on $L^2(\mathfrak{E})$ are unitary. Using this fact and setting $f=\Theta_1^* u$, it follows from (1) that $\Theta_2 \Lambda f=\Delta\Theta_1 f=\Delta u\in\Theta_2 H^2(\mathfrak{E})$, and hence $\Lambda f\in H^2(\mathfrak{E})$. On the other hand, $\Theta_1 f=\Theta_1\Theta_1^* u=u$. Thus, Λf and $\Theta_1 f$ are both in $H^2(\mathfrak{E})$. From this it follows that

$$(\det\Lambda)f\in H^2(\mathfrak{E}) \quad\text{and}\quad (\det\Theta_1)f\in H^2(\mathfrak{E}).$$

Since $\det\Lambda$ and $\det\Theta_1$ are relatively prime, we infer by using a lemma of Sz.-Nagy [9, p. 74], that $f\in H^2(\mathfrak{E})$. This implies $u\in\Theta_1 H^2(\mathfrak{E})$. As $u\in\mathfrak{H}(\Theta_1)$, we necessarily have $u=0$. Hence, X is one to one, and the proof of the theorem is complete.

It is shown in [6], Theorem 3. 1 that every $N\times N$ matrix over H^∞ is quasi-equivalent to a diagonal one with the invariant factors on the main diagonal. Thus if Θ is the characteristic operator function of an operator T of class $C_0(N)$, then Θ is quasi-equivalent to a normal matrix Θ', i.e. Θ' is diagonal and the diagonal entries of Θ' are the invariant factors of Θ. From the theorem then we have that T is quasi-similar to $S(\Theta')$. Operators of the form $S(\Theta')$ are called Jordan operators by Sz.-Nagy and Foiaş, and they were the first to show that every $C_0(N)$ contraction T is quasi-similar to a Jordan operator and that the minimal inner function of T is the first invariant factor of Θ_T [11, 15]. We have thus obtained their results via a different route; moreover, we have shown that the inner functions that appear in the Jordan model of an operator are related to the characteristic operator function in the manner they conjectured. To summarize, we have established the following:

Theorem 3. *If T is an operator of class $C_0(N)$ for some integer N, then T is quasi-similar to a Jordan operator determined by the invariant factors of the characteristic operator function of T.*

Finally, Theorem 2 has a converse; that the statement is more general is illusory.

Theorem 4. *Let $\mathfrak{E}$ be finite dimensional and suppose that Θ_1 and Θ_2 are inner functions that induce isometries on $H^2(\mathfrak{E})$. If $S(\Theta_1)$ is a quasi-affine transform of $S(\Theta_2)$, then Θ_1 and Θ_2 are quasi-equivalent.*

Proof. Let Θ_1' and Θ_2' be the normal matrices that are quasi-equivalent to Θ_1 and Θ_2 respectively. Then by Theorem 2 and the hypothesis,

$$S(\Theta_1')\prec S(\Theta_1)\prec S(\Theta_2)\prec S(\Theta_2').$$

It was shown by Sz.-Nagy and Foiaş [14] that if one Jordan operator is a quasi-affine transform of another, then they are both determined by the same nonconstant inner functions. Consequently, $S(\Theta_1')=S(\Theta_2')$, and hence $\Theta_1'=\Theta_2'$. It follows by transitivity that Θ_1 and Θ_2 are quasi-equivalent.

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