On quasi-equivalence of matrices over H^∞

By ERIC A. NORDGREN in Durham (New Hampshire, U.S.A.)

Dedicated to Professzor Béla Szőkefalvi-Nagy on his 60th birthday

Introduction

The purpose of this paper is to introduce a relation of quasi-equivalence for matrices over H^∞ that generalizes the relation of equivalence for matrices over principal ideal domains (cf. [3], p. 79) and leads to an analogous theory. In [6] Sz.-Nagy and Foiaş began a study of a class C_0 of Hilbert space contractions that possess a minimal function analogous to the minimal polynomial of finite matrices. This study was continued in [7] where it was shown that the minimal function of a C_0 contraction T of finite defect bears the same relation to the characteristic operator function Θ_T of T that the minimal polynomial of a finite matrix A bears to the polynomial matrix $A-\lambda$. In this paper an equivalence theory is developed which will be used in a subsequent paper [4] to show that the invariant factors of Θ_T determine the Jordan model of T, which was introduced by Sz.-Nagy and Foiaş in [8]. Thus the analogy between such contractions and finite matrices is complete.

1. Preliminaries

We will be concerned with matrices over the Hardy class H^∞ of bounded analytic functions on the unit disc, and a few of the pertinent facts will be set forth here. See [2] or [9] for details. Since H^∞ is an integral domain, the usual terminology for factorization applies. In particular, for a, b in H^∞, a is said to divide b if there exists c in H^∞ such that $ac=b$, in which case we will write $a|b$. According to Fatou's theorem, every H^∞ function has a radial limit at almost every point of the unit circle, and if these radial limits have modulus one almost everywhere, then the function is called inner. Every H^∞ function $f\not\equiv 0$ can moreover be factored into an inner function f_i and a function f_e having only constant inner divisors. We will require

[1]) Research supported in part by a grant from the National Science Foundation.

that the first nonvanishing Taylor coefficient of f_i be positive, and in this case the factorization is unique. The inner part of f can be further factored into a Blaschke product b_f, determined by the zeros of f, and a singular inner function s_f, determined by a measure ν_f on the unit circle that is singular with respect to Lebesgue measure. An inner function g divides f if and only if every zero of g is a zero of f, counting multiplicity, and $\nu_g \leqq \nu_f$.

Every subset Φ of H^∞ has a greatest common divisor $\wedge \Phi$, i.e. an inner function that divides every member of Φ and is in turn divisible by every other inner function with this property. We will require that $\wedge \Phi = (\wedge \Phi)_i$, thus insuring uniqueness. For pairs f, g, we will write $f \wedge g$ simply in place of $\wedge \{f, g\}$. A subset Φ of H^∞ is *relatively prime* if $\wedge \Phi = 1$. If Φ is any subset of H^∞, then let $\Phi^k = \{f^k : f \in \Phi\}$.

Lemma 1. 1. *If Φ is relatively prime, then so is Φ^k for every positive integer k. If Ψ is also relatively prime and if*

$$\Phi\Psi = \{\varphi\psi : \varphi \in \Phi, \psi \in \Psi\},$$

then $\Phi\Psi$ is relatively prime.

Proof. The members of Φ have no common zero, and the same is true of Φ^k. Thus no nonconstant Blaschke product divides each member of Φ^k. If ν is any nonzero singular measure, then there is an f in Φ such that $\frac{1}{k}\nu$ is not dominated by ν_f. Consequently, if $g = f^k$, then g^k is in Φ^k and ν is not dominated by $k\nu_f = \nu_g$. Thus no nonconstant singular inner function divides every member of Φ^k, and hence Φ^k is relatively prime.

If g is inner and if $g|\varphi\psi$ for all φ in Φ and ψ in Ψ, then $g|\psi$ since Φ is relatively prime, and it follows that g is constant since Ψ is relatively prime.

A notion of length can be attached to elements of a principal ideal domain, and this idea can be used to show that any matrix over such a ring can be reduced to a diagonal one by a finite number of equivalence transformations. For H^∞ a different route to diagonalization is available because of the possibility of forming a Lebesgue decomposition of one inner function with respect to another. If f and g are inner functions, if every zero of f is a zero of g, and if $\nu_f \ll \nu_g$, then we will write $f \ll g$. On the other hand, if f and g have no common zeros and if $\nu_f \perp \nu_g$, then we will write $f \perp g$. It is easy to see that $f \perp g$ if and only if $f \wedge g = 1$. Suppose f and g are arbitrary inner functions. Then $f = f_a f_s$, where $f_a \ll g$ and $f_s \perp g$. For let $\nu_f = \nu_a + \nu_s$ be the Lebesgue decomposition of ν_f with respect to ν_g. Let each zero of f that is a zero of g be a zero of f_a, and let ν_a determine the singular inner factor of f_a. Let the remaining zeros of f be zeros of f_s, and let ν_s determine the singular part of f_s. Then the desired factorization of f results. The essential lemma for the diagonalization later is the following.

Lemma 1. 2. *If a and b are relatively prime H^∞ functions, and if ω and ψ are arbitrary inner functions, then there exists H^∞ functions x and y such that $y\wedge\omega = 1$ and $(ax+by)\wedge\psi = 1$.*

Proof.[2]) Let $\omega=\omega_1\omega_2\omega_3$, where $\omega_1\ll a_i$, $\omega_2\ll b_i$, and $\omega_3\perp a_i b_i$. Setting $a'=a\omega_3$ we have $a'_i\perp b_i$. Now factor ψ in the form $\psi=\psi_1\psi_2\psi_3$, where $\psi_1\ll a'_i$, $\psi_2\ll b_i$, and $\psi_3\perp a'_i b_i(=a_i b_i\omega_3)$; hence $\psi_3\perp\omega$. Set $x'=\psi_1\psi_2+\psi_3$, $y=\psi_3$, and $\delta=a'x'+by$. Clearly $y\wedge\omega=1$. We shall also show that $\delta\wedge\psi=1$.

Consider to this effect any inner divisor φ of $\delta\wedge\psi$. Since $\varphi|\psi$, we have $\varphi=\varphi_1\varphi_2\varphi_3$, where $\varphi_k=\varphi\wedge\psi_k$ $(k=1, 2, 3)$. Set $\hat\varphi_1=\varphi_1\wedge a'_i$, $\hat\varphi_2=\varphi_2\wedge b_i$, and observe that φ_1 is constant if $\hat\varphi_1$ is so, and similarly for φ_2 and $\hat\varphi_2$. Since $\hat\varphi_1|\delta$ and $\hat\varphi_1|a'$, we have $\hat\varphi_1|by$, and as $a'\perp b$ we deduce that $\hat\varphi_1|y$, i.e. $\hat\varphi_1|\psi_3$. But $\hat\varphi_1|\varphi_1|\psi_1$ and $\psi_1\perp\psi_3$ so $\hat\varphi_1$ — and therefore φ_1 — are constant. Similarly, from $\hat\varphi_2|\delta$ and $\hat\varphi_2|b$ we deduce that $\hat\varphi_2|a'x'$ and as $a'\perp b$ we conclude that $\hat\varphi_2|x'(=\psi_1\psi_2+\psi_3)$. But $\hat\varphi_2|\varphi_2|\psi_2$ and $\psi_2\perp\psi_3$ so $\hat\varphi_2$ — and therefore φ_2 — are constant. Thus $\varphi(=\varphi_1\varphi_2\varphi_3)|\psi_3$. On the other hand, we have $\varphi|\delta(=a'\psi_1\psi_2+a'\psi_3+b\psi_3)$, and hence $\varphi|a'\psi_1\psi_2$. As the factors of the last product are prime to ψ_3, φ is constant. This proves that $\delta\wedge\psi=1$. To obtain x as required by the lemma, we only have to set $x=\omega_3 x'$.

2. Definition and elementary properties of quasi-equivalence

If A and B are $m\times n$ matrices over H^∞, then equivalence of A and B is defined by requiring the existence of units X and Y of orders m and n respectively such that $XA=BY$. Here a unit X of order m is an $m\times m$ matrix over H^∞ for which there exists another such matrix Z such that $XZ=ZX=I_m$, where I_m is the $m\times m$ identity matrix. Since only weak* closed ideals in H^∞ are principal [5], this is not the appropriate relation to study if one hopes to obtain a theory analogous to the classical one, as may be seen from the following example. Suppose a and b are relatively prime inner functions, and let $A=\operatorname{diag}(a, b)$, $B=\operatorname{diag}(ab, 1)$. A calculation shows that A and B are equivalent only if there exist x and y in H^∞ such that $ax+by=1$, which implies that a and b have to satisfy the Carleson condition in addition to being relatively prime.

A *quasi-unit* $\mathbf{X}$ of order n is a collection of $n\times n$ matrices over H^∞ such that $\det\mathbf{X}$ is relatively prime, where $\det\mathbf{X}=\{\det X: X\in\mathbf{X}\}$. Clearly, if X is a unit, then $\{X\}$ is a quasi-unit, but the collection of all nonconstant inner functions a quasi-unit of order one which contains no unit. It is easy to see that products of quasi-units

[2]) I am indebted to the Referee for supplying the present version of the proof, which is much more lucid than the original.

are also quasi-units: if $\mathbf{X}$ and $\mathbf{Y}$ are quasi-units of the same order, then $\mathbf{XY}=\{XY: X\in\mathbf{X}$ and $Y\in\mathbf{Y}\}$ is also a quasi-unit since $\det \mathbf{XY} = \det \mathbf{X}\cdot\det \mathbf{Y}$, which is relatively prime by Lemma 1. 1.

If A and B are $m\times n$ matrices over H^∞, then A will be called *quasi-equivalent* to B if there exist quasi-units $\mathbf{X}$ and $\mathbf{Y}$ of orders m and n respectively such that $\mathbf{X}A=B\mathbf{Y}$. By the remarks of the preceding paragraph, equivalence implies quasi-equivalence, and quasi-equivalence is transitive.

In presenting some of our arguments the following definition will be found useful. If A and B are $m\times n$ matrices and δ is an H^∞ function, then A will be called *δ-equivalent* to B if there are square matrices X and Y of orders m and n respectively such that $XA=BY$ and $(\det X)_i$ and $(\det Y)_i$ are factors of δ. It is immediate that if A is δ-equivalent to B for all δ in a relatively prime family, then A is quasi-equivalent to B. Let A^t denote the transpose of A.

Lemma 2. 1.

a) *If A is δ-equivalent to B and B is ε-equivalent to C, then A is $\delta\varepsilon$-equivalent to C.*

b) *If A is δ-equivalent to B, then B is $\delta^{k(k-1)}$-equivalent to A, where k is the larger of the dimensions of A and B.*

c) *If A is δ-equivalent to B, then A^t is $\delta^{k(k-1)}$-equivalent to B^t, where k is as above.*

Proof.

a) If $XA=BY$ and $UB=CV$, then $UXA=CVY$, and the assertion follows from the multiplicative property of determinants.

b) If $XA=BY$, then multiplying this equation on the left by adj X, the classical adjoint of X, and on the right by adj Y leads to

$$(\det Y)(\operatorname{adj} X)B=A(\det X)(\operatorname{adj} Y).$$

If X is $m\times m$, then

$$\det(\det Y\cdot \operatorname{adj} X)=(\det Y)^m(\det X)^{m-1},$$

and this together with the corresponding relation for $\det X\cdot\operatorname{adj} Y$ implies the assertion.

c) This part follows from the defining relation for δ-equivalence by taking transposes and applying part b).

Invariant factors for matrices over H^∞ may be defined in the usual way. If A is an $m\times n$ matrix let $\mathscr{D}_0(A)=1$ and let $\mathscr{D}_k(A)$ be the greatest common divisor of all minors of order k of A, where k is no larger than $\min\{m, n\}$. The invariant factors are then defined by $\mathscr{E}_k(A)=\mathscr{D}_k(A)/\mathscr{D}_{k-1}(A)$ for $k\geqq 1$ such that $\mathscr{D}_k(A)\neq 0$.

Lemma 2. 2. *If A is δ-equivalent to B, then $\mathcal{D}_k(A)|\delta^k\mathcal{D}_k(B)$ and $\mathcal{D}_k(B)|\delta^k\mathcal{D}_k(A)$ for all k such that $\mathcal{D}_k(A)\neq 0$.*

Proof. Suppose $XA=BY$, $(\det X)_i|\delta$ and $(\det Y)_i|\delta$. From the fact that the minors of a product of matrices are linear combinations of the minors of corresponding order of either factor, it follows that $\mathcal{D}_k(A)|\mathcal{D}_k(XA)$ and also $\mathcal{D}_k(BY)|\mathcal{D}_k((\det Y)B)$, since $(\det Y)B=BY \operatorname{adj} Y$. By supposition, $\mathcal{D}_k(XA)=\mathcal{D}_k(BY)$, and hence

$$\mathcal{D}_k(A)\,|\mathcal{D}_k((\det Y)B),\quad \text{i.e.}\quad \mathcal{D}_k(A)|(\det Y)^k\mathcal{D}_k(B).$$

This implies $\mathcal{D}_k(A)|\delta^k\mathcal{D}_k(B)$, and the other relation may be obtained similarly.

Theorem 2. 1. *If two matrices over H^∞ are quasi-equivalent, then they have the same invariant factors.*

Proof. Suppose A and B are matrices over H^∞, and $\mathbf{X}$ and $\mathbf{Y}$ are quasi-units such that $\mathbf{X}A=B\mathbf{Y}$. If $XA=BY$, then as in the proof of Lemma 2. 2, $\mathcal{D}_k(A)|(\det Y)^k\mathcal{D}_k(B)$. Since $\mathbf{Y}$ is a quasi-unit, it follows from Lemma 1. 1 that $\mathcal{D}_k(A)|\mathcal{D}_k(B)$. The relation $\mathcal{D}_k(B)|\mathcal{D}_k(A)$ follows similarly, and hence $\mathcal{D}_k(A)=\mathcal{D}_k(B)$ which implies the assertion.

3. Diagonalization

Our principal goal is to prove the converse of Theorem 2. 1, and this will be accomplished by showing that every matrix is quasi-equivalent to a canonical one. A matrix E over H^∞ is in *normal form* (or simply *normal*) provided

$$E=\begin{bmatrix} D & 0\\ 0 & 0\end{bmatrix},$$

where D is a diagonal matrix of nonzero inner functions, each with a positive first nonvanishing Taylor coefficient, and each one except the first divisible by its predecessor. (Some of the blocks of zeros or even D may not be present.) As in the classical case, the diagonal entries of D are the invariant factors of E (see e.g. [3], p. 91).

Lemma 3. 1. *If $A=\begin{bmatrix} A_{11} & 0\\ A_{21} & A_{22}\end{bmatrix}$, then for each inner function ψ there is a matrix X such that* $(\det X)\wedge\psi=1$ *and AX is of the form $\begin{bmatrix} B_{11} & 0\\ B_{21} & B_{22}\end{bmatrix}$, where $B_{11}=A_{11}$, and except possibly for the first, all entries of the first row of B_{22} are zeros.*

Proof. Let $(a_1, a_2, \ldots, a_k)$ be the first row of A_{22}. It will suffice to produce a $k\times k$ matrix X whose determinant is relatively prime to ψ such that $(a_1, a_2, \ldots, a_k)X$ has at most a nonvanishing first entry. For the required matrix may then be produced by forming the direct sum of an appropriate identity matrix with X.

The matrix X is obtained in $k-1$ steps, each step changing one a_j to zero. If the a_j are all zero, then there is nothing to prove. Permuting columns if necessary, assume $a_1 \neq 0$ and let $\omega = a_1 \wedge a_2$. By Lemma 1. 2, x and y may be chosen so that if $\delta_1 = (a_1 x + a_2 y)/\omega$, then $\delta_1 \wedge \psi = 1$. Let

$$X_1 = \begin{bmatrix} x & -a_2/\omega \\ y & a_1/\omega \end{bmatrix} \oplus I_{k-2}.$$

Then $\det X_1 = \delta_1$, the second component of $(a_1, a_2, \ldots, a_k)X_1$ is zero, and the entries beyond the second are unchanged.

After the second and third columns of the result are permuted this procedure may be repeated, and in $k-1$ steps a matrix is produced that has its only nonzero entry as the first. The matrix X is obtained as a product of permutation matrices and matrices of the form X_1, and the result follows from the fact that products of functions relatively prime to ψ are also relatively prime to ψ.

Lemma 3. 2. *If A is an $m \times n$ matrix over H^∞ and ψ is any inner function then there exist an $m \times m$ matrix X and an $n \times n$ matrix Y, each with determinant relatively prime to ψ, such that XA is upper triangular and AY is lower triangular.*

Proof. The upper triangular case follows from the lower by taking transposes, and the lower triangular case is proved by repeated use of Lemma 3. 1.

Theorem 3. 1. *Every finite matrix over H^∞ is quasi-equivalent to a unique normal matrix. In fact given any $m \times n$ matrix A over H^∞ and any inner function ψ, A is δ-equivalent to the normal matrix formed from the invariant factors of A for some δ relatively prime to ψ.*

Proof. Since a normal matrix is determined by its invariant factors, uniqueness is a consequence of Theorem 2. 1. The second statement implies the remaining part of the first, and by Lemma 2. 1. c, it suffices to consider the case $m \leqq n$, since the case $m > n$ follows by taking transposes.

Given an inner function ψ, if δ is relatively prime to an inner multiple of ψ, then it is also relatively prime to ψ. Hence there is no loss of generality in considering a ψ divisible by each nonzero $\mathcal{D}_k(A)$. We suppose further without loss of generality that $\mathcal{D}_1(A) = 1$. The major portion of the proof consists of verifying that there exists η relatively prime to ψ such that A is η-equivalent to a normal matrix E_1.

The proof is by induction on m, and the case $m=1$ (and arbitrary $n \geqq 1$) is an easy consequence of Lemma 3. 2. For it implies the existence of an $n \times n$ matrix Y having a determinant relatively prime to ψ such that AY is lower triangular, i.e. AY is a $1 \times n$ matrix with at most its first entry a nonzero. Thus $AY = XE_1$, where X is the 1×1 matrix whose single entry is the outer factor of a, and E_1 is the $1 \times n$

normal matrix whose first entry is the inner factor of a. Taking $\eta=(\det Y)^{n(n-1)}$, we see by Lemma 2. 1 that A is η-equivalent to E_1.

Suppose the assertion true, therefore, for $(m-1)\times v$ matrices with $v \geqq m-1$. By Lemmas 3. 2 and 2. 1 again, there exists a δ_1, relatively prime to ψ such that A is δ_1-equivalent to a lower triangular matrix A_1. The last $n-m$ columns of A_1, which consist only of zeros, do not essentially affect the subsequent calculations, and thus it will be assumed that A_1 is an $m\times m$ lower triangular matrix.

If A_1' consists of the last $m-1$ rows and columns of A_1, then the inductive hypothesis implies there exists a δ_2 relatively prime to ψ such that A_1' is δ_2-equivalent to a normal matrix E_1', the equivalence being effected by a pair of matrices X_1' and Y_1'. If A_1'' consists of the last $m-1$ rows of the first column of A_1, then let A_2 have the same first row as A_1 and $(X_1' A_1'' E_1')$ as its last $m-1$ rows. Let $X_1 = I_1\oplus X_1'$ and $Y_1 = I_1\oplus Y_1'$. Then $X_1A_1=A_2Y_1$, i.e. A_1 is δ_2-equivalent to A_2, and $\mathscr{D}_1(A_2)$ is the greatest common divisor of the entries in the first two columns of A_2.

By Lemma 3. 2, there exists X_2 with determinant δ_3 relatively prime to ψ such that if $A_3'=X_2A_2$, then A_3' is upper triangular. The greatest common divisor ε of the elements in the first two columns of A_3' is a factor of $\delta_3\mathscr{D}_1(A_2)$, as may be seen by applying Lemma 2. 2 to the first two columns of A_2 and A_3'. But A is $\delta_1\delta_2$-equivalent to A_2 by Lemma 2. 1. a, and thus Lemma 2. 2 together with the initial supposition on $\mathscr{D}_1(A)$ yield $\varepsilon|\delta_1\delta_2\delta_3$. Hence if A_3 is obtained from A_3' by dividing the entries in the first two columns by ε, then A_3' is ε^2-equivalent to A_3 and $\varepsilon^2\wedge\psi = 1$. Further, if a and b are the first two entries of the first row of A_3 and if c is the second entry of the second row, then $\wedge\{a, b, c\}=1$.

It may be assumed that a or b is nonzero, for otherwise the interchange of the first two rows and columns yields an equivalent matrix satisfying this condition. Let $\omega = a\wedge b$, and choose x and y in H^∞ according to Lemma 1. 2 so that $y\wedge\omega = 1$ and if $\delta_4 = (ax+by(/\omega$, then $\delta_4\wedge\psi = 1$. If

$$X_2' = \begin{bmatrix} x & -b/\omega \\ y & a/\omega \end{bmatrix} \oplus\delta_4 I_{m-2},$$

then all entries of A_3X_2', except possibly for those in the second row, are divisible by δ_4, and hence if $Y_2=\operatorname{diag}(\delta_4, 1, \delta_4, \delta_4, \ldots, \delta_4)$, then $A_3X_2'=Y_2A_4$, where the first two entries ω and cy of the first column of A_4 are relatively prime. From the form of X_2' and Y_2 it is not hard to see that A_3 is δ_4-equivalent to A_4.

Since ω and cy are relatively prime, another application of Lemma 1. 2 yields H^∞ functions u and v such that $\delta_5 = u\omega+vcy$ is relatively prime to ψ. Let

$$X_3 = \begin{bmatrix} u & v \\ -cy & \omega \end{bmatrix} \oplus I_{m-2};$$

X_3 has determinant δ_5, and X_3A_4 has δ_5 as the only nonzero entry of the first column.

If A_5 is obtained from $X_3 A_4$ be replacing this entry by 1, and if $Y_3 \doteq (\delta_5) \oplus I_{m-1}$, then $X_3 A_4 = A_5 Y_3$. By equivalence transformations, all entries but the first of row one of A_5 can be changed to zeros yielding an equivalent matrix A_6 that is a direct sum of (1) with an $(m-1)\times(m-1)$ matrix. A second application of the induction hypothesis then yields δ_6 relatively prime to ψ such that A_6 is δ_6-equivalent to a normal matrix E_1.

Combining the above steps, we see that if η is the product of the six δ_j's and ε^2, then η is relatively prime to ψ and A is η-equivalent to a normal matrix E_1. This completes the induction.

In general it can not be supposed that E_1 is the matrix E formed from the invariant factors of A. By Lemma 2. 2, however, there exist inner functions $\alpha_1, \alpha_2, \ldots, \alpha_k$, where k is the largest of the indices for which $\mathscr{D}_j(A) \neq 0$, such that each α_j divides η^j and

$$\mathscr{D}_j(E_1) = \alpha_j \mathscr{D}_j(A).$$

Since each $\mathscr{D}_j(A)$ divides ψ, and since η is prime to ψ, it follows that α_j is prime to $\mathscr{D}_l(A)$ for all j and l. Thus with $\alpha_0 = 1$,

$$\mathscr{E}_j(E_1) = (\alpha_j/\alpha_{j-1})\mathscr{E}_j(A),$$

and each α_j/α_{j-1} is inner. Thus if

$$Y_4 = \operatorname{diag}(\alpha_1/\alpha_0, \alpha_2/\alpha_1, \ldots, \alpha_k/\alpha_{k-1}) \oplus I_{n-k},$$

then $\det Y_4 = \alpha_k$, which divides η, and $E_1 = EY_4$. Hence if $\delta = \eta^2$, then $\delta \wedge \psi = 1$ and A is δ-equivalent to E. On the other hand,

$$\mathscr{E}_j(A)\alpha_j^2 \mathscr{E}_{j+1}(E_1)/\mathscr{E}_j(E_1) = \alpha_{j+1}\alpha_{j-1}\mathscr{E}_{j+1}(A) \quad (j = 1, \ldots, k-1),$$

and this together with the fact that $\mathscr{E}_j(A)$ is relatively prime to $\alpha_{j+1}\alpha_{j-1}$ imply that $\mathscr{E}_j(A) | \mathscr{E}_{j+1}(A)$, i. e. E is normal. This completes the proof.

Corollary 3. 1. *An $m\times n$ matrix A is quasi-equivalent to an $m\times n$ matrix B over H^∞ if and only if A and B have the same invariant factors.*

Proof. Necessity was established in Theorem 2. 1. If A and B have the same invariant factors, then each one determines the same normal matrix E. By the theorem, A is quasi-equivalent to E which is quasi-equivalent to B, and this establishes the result.

Corollary 3. 2. *Quasi-equivalence is an equivalence relation.*

Corollary 3. 3. *If A and B are quasi-equivalent, then there exist matrices X and Y each of whose determinants is relatively prime to all the invariant factors of A and B, and such that $XA = BY$.*

Proof. Let E be the normal matrix that A and B are quasi-equivalent to, and let ψ be a multiple of each of the nonzero entries of E. Two applications of the theorem yield δ_1 and δ_2 relatively prime to ψ such that A is δ_1-equivalent to E and B is δ_2-equivalent to E. If k is the larger of m and n, then setting $\delta=\delta_1\delta_2^{k(k-1)}$, we see from Lemma 2. 1 that A is δ-equivalent to B. Thus there exist X and Y such that $(\det X)_i|\delta$, $(\det Y)_i|\delta$, (hence $\det X$ and $\det Y$ are relatively prime to ψ) and $XA=BY$.

Corollary 3. 4. *Suppose $XA=BY$. If the determinants of X and Y are relatively prime to the invariant factors of A and B, then A and B are quasi-equivalent.*

Proof. The first hypothesis implies that A and B are δ-equivalent for $\delta = (\det X\cdot\det Y)_i$, By Lemma 2. 2, $\mathscr{D}_k(A)|\delta^k\mathscr{D}_k(B)$, and by the second hypothesis, $\mathscr{D}_k(A)|\mathscr{D}_k(B)$. Similarly $\mathscr{D}_k(B)|\mathscr{D}_k(A)$, and hence A and B have the same invariant factors. Thus Corollary 3. 1 implies the result.

Corollary 3. 5. *Suppose $XA=BY$, where A and B are square matrices. If X and Y have the same determinant, and it is relatively prime to* $\det A$, *then A and B are quasi-equivalent.*

Proof. If $\det A=0$, then $\det X$ and $\det Y$ are outer and hence relatively to prime to the invariant factors of A and B. If $\det A\neq 0$, then the relation $XA=BY$ implies that $\det B=\det A$. Since $\det A$ is the product of the invariant factors of A up to an outer factor, it follows that $\det X$ and $\det Y$ are relatively prime to the invariant factors of A, and similarly to those of B. Hence in either case the result follows from Corollary 3. 4.

4. A reformulation

The definition of quasi-equivalence may be formulated in a slightly different way which is more general and leads to an open question. Let $\mathfrak{H}$ be a separable Hilbert space and suppose X is a bounded analytic function on the unit disc D whose values are operators on $\mathfrak{H}$ (see [1] or [7], Chap. V.). If X admits a scalar multiple, let Φ_X be the set of scalar multiples of X, and let $\varphi_X=\wedge\Phi_X$. Then in the case of X inner φ_X is the characteristic scalar inner function of X (cf. [1], p. 81). If $\mathbf{X}$ is a collection of analytic operator valued functions admitting scalar multiples such that $\{\varphi_X : X\in\mathbf{X}\}$ is relatively prime, then $\mathbf{X}$ is called a quasi-unit on $\mathfrak{H}$. In the case of $\mathfrak{H}$ finite dimensional, this definition agrees with the one given previously ([1], p. 81; [7]).

Let $\mathfrak{H}$ and $\mathfrak{K}$ be a pair of separable Hilbert spaces. If A and B are bounded analytic functions on D whose values are operators from $\mathfrak{H}$ to $\mathfrak{K}$, then A is quasi-equivalent to B in case there exist quasi-units $\mathbf{X}$ and $\mathbf{Y}$ on $\mathfrak{H}$ and $\mathfrak{K}$ respectively such that $\mathbf{X}A=B\mathbf{Y}$. Is every operator function or every operator function that admits a scalar multiple quasi-equivalent to a diagonal one?

References

[1] H. HELSON, *Lectures on invariant subspaces*, Academic Press (New York, 1964).
[2] K. HOFFMAN, *Banach spaces of analytic functions*, Prentice Hall (Englewood Cliffs, N. J., 1962).
[3] J. JACOBSON, *Lectures in abstract algebra*, Vol. II, Van Nostrand (Princeton, N. J., 1953).
[4] B. MOORE, III and E. A. NORDGREN, On quasi-equivalence and quasi-similarity, *Acta Sci. Math.*, **34** (1973), 283—288.
[5] T. P. SRINIVASAN, Simply invariant subspaces and generalized analytic functions, *Proc. Amer. Math. Soc.*, **16** (1965), 813—818.
[6] B. SZ.-NAGY and C. FOIAŞ, Sur les contractions de l'espace de Hilbert. VII. Triangulations canoniques. Fonction minimum, *Acta Sci. Math.*, **25** (1964), 12—37.
[7] B. SZ.-NAGY and C. FOIAŞ, Sur les contractions de l'espace de Hilbert. VIII Fonctions caractéristiques. Modèles fonctionnels, *Acta Sci. Math.*, **25** (1964), 38—71.
[8] B. SZ.-NAGY and C. FOIAŞ, Modèle de Jordan pour une classe d'opérateurs de l'espace de Hilbert, *Acta Sci. Math.*, **31** (1970), 93—117.
[9] B. SZ.-NAGY and C. FOIAŞ, *Harmonic analysis of operators on Hilbert space*, Akadémiai Kiadó (Budapest, 1970).

(Received August 12, 1971)