On commutative universal algebras

By L. KLUKOVITS in Szeged

To Professor Béla Szőkefalvi-Nagy on his 60th birthday

Universal algebras are called commutative (or Abelian) in [2], [3] and [4] under different conditions. It has to be remarked that in the case of groups these conditions are satisfied only for the Abelian groups and this is the reason why the term "commutative" is used. The conditions mentioned above are the following:

M. (B. I. PLOTKIN [2], p. 32.) In case of a universal algebra (shortly algebra) (A, Ω) for any operations μ and ν (m- and n-ary ones, respectively) the equality $(a_{11} \dots a_{1m}\mu) \dots$ $\dots (a_{n1} \dots a_{nm}\mu)\nu = (a_{11} \dots a_{n1}\nu) \dots (a_{1m} \dots a_{nm}\nu)\mu$ holds under any matrix $(a_{ij})_{n \times m}$ over A (see also P. M. COHN [1], p. 127).

H. (A. G. KUROŠ [3], p. 92.) The set of all homomorphisms from any algebra (G, Ω) into the algebra (A, Ω) admits the operations in Ω, i.e., if $\varphi_1, \dots, \varphi_m$ are homomorphisms from (G, Ω) into (A, Ω) and $\mu (\in \Omega)$ is any m-ary operation, then the mapping

$$(\varphi_1 \dots \varphi_m \mu): G \to A$$

defined for any element $g \in G$ as

$$g(\varphi_1 \dots \varphi_m \mu) = (g\varphi_1) \dots (g\varphi_m)\mu$$

is also a homomorphism. It has to be mentioned that in [3], the case of 0-ary operation was studied separately. However, this fact is irrelevant here, since an operation of this type can be considered a special unary operation ω with $a\omega = b\omega$ for any elements a, b. Thus we can suppose that in the set Ω there are no 0-ary operations.

E. (B. CSÁKÁNY [4].) The set of all endomorphisms of the algebra (A, Ω) admits the operations in Ω, i.e., for any endomorphisms $\varepsilon_1, \dots, \varepsilon_m$ and any m-ary operation μ (in Ω) the mapping

$$(\varepsilon_1 \dots \varepsilon_m \mu): A \to A$$

defined for any element $a \in A$ as

$$a(\varepsilon_1 \dots \varepsilon_m \mu) = (a\varepsilon_1) \dots (a\varepsilon_m)\mu$$

is an endomorphism, too.

On the basis of a notion due to M. Servi [5] we introduce a fourth condition. Consider any category $\mathscr{C}$, admitting finite direct composition. For any object $A(\in \mathrm{Ob}\,\mathscr{C})$, each morphism $\varphi \in \mathrm{Hom}\,(A^m, A)$ (m is a natural number) is called m-ary operation on A. If $A \in \mathrm{Ob}\,\mathscr{C}$ and Φ is a set of operations defined above, let the ordered couple (A, Φ) be called a Servi-algebra in the given category $\mathscr{C}$ (e.g., the topological algebras are exactly the Servi-algebras in the category of all topological spaces).

So the fourth condition is as follows:

C. The algebra (A, Ω) is a Servi-algebra in the category of algebras of the same type $\mathscr{A}$.

We prove that these conditions are equivalent for varieties of algebras.

Lemma. *For any universal algebra (A, Ω) the conditions* **M**, **H** *and* **C** *are equivalent and* **E** *follows from each of them.*

Proof. The equivalence of conditions **M** and **H** is proved in [3], therefore we have only to prove the equivalence of **M** and **C**.

Consider any category $\mathscr{A}$ of universal algebras similar to a given algebra (A, Ω). We want to prove that the mapping

$$\bar{\mu}: A^m \to A$$

defined by

$$(x_1, \ldots, x_m)\,\bar{\mu} = x_1 \ldots x_m\,\mu$$

is a homomorphism from (A^m, Ω) into (A, Ω). Let $(a_{11}, \ldots, a_{1m}), \ldots, (a_{n1}, \ldots, a_{nm}) \in A^m$, and let ν be any n-ary operation in Ω. Then we have

$$\big((a_{11}, \ldots, a_{1m}) \ldots (a_{n1}, \ldots, a_{nm})\,\nu\big)\,\bar{\mu} = \big((a_{11} \ldots a_{n1}\,\nu), \ldots, (a_{1m} \ldots a_{nm}\,\nu)\big)\,\bar{\mu} =$$
$$= (a_{11} \ldots a_{n1}\,\nu) \ldots (a_{1m} \ldots a_{nm}\,\nu)\,\mu,$$

and, on the other hand, we have

$$\big((a_{11}, \ldots, a_{1m})\,\bar{\mu}\big) \ldots \big((a_{n1}, \ldots, a_{nm})\,\bar{\mu}\big)\,\nu = (a_{11} \ldots a_{1m}\,\mu) \ldots (a_{n1} \ldots a_{nm}\,\mu)\,\nu.$$

Now, if condition **M** holds, then we can see that $\bar{\mu}$ is a homomorphism and conversely, if $\bar{\mu}$ is a homomorphism, then condition **M** holds true. The second statement of the lemma is obvious.

A simple counter-example shows that, in general from the condition **E** the condition **M** does not follow. In fact let us consider the groupoid G, defined by the following multiplication table:

γ	a	b	c
a	a	b	c
b	b	a	b
c	c	c	a

G has three endomorphisms given by the following table

	ω	ι	α
a	a	a	a
b	a	b	c
c	a	c	b

Hence it is obvious that condition **E** holds for G. On the other hand,

$$(ab\gamma)(ca\gamma)\gamma = b, \qquad (ac\gamma)(ba\gamma)\gamma = c,$$

i.e., condition **M** is not valid.

We shall say that for a variety of universal algebras the conditions mentioned above hold if they hold for each algebra in the given variety.

Theorem. *For any variety of universal algebras all the conditions* **M**, **H**, **E** *and* **C** *are equivalent.*

Proof. In view of the lemma, it is enough to prove if condition **E** holds for the variety $\mathfrak{A}$, then condition **M** holds, too. The way of proving this is similar to that of T. Evans [6] concerning groupoids.

Let F denote the free algebra generated by the set $X=\langle x_1, x_2, \dots\rangle$ in the variety $\mathfrak{A}$. It is sufficient to show that the equality in condition **M** holds for any $n\times m$ matrix over X.

Let μ and ν be arbitrary operations (m- and n-ary respectively) in the variety $\mathfrak{A}$ and $(x_{ij})_{n\times m}$ any matrix over X. We define the following mappings ε_k ($k=1, 2, \dots, m$) on the set X,

$$x_{j1}\varepsilon_k = x_{jk}$$

for all $1\leqq j\leqq n$. These mappings can be extended to endomorphisms of the free algebra F. Thus we have

$$x_{j1}(\varepsilon_1 \dots \varepsilon_m \mu) = (x_{j1}\varepsilon_1)(x_{j1}\varepsilon_2)\dots(x_{j1}\varepsilon_m)\mu = x_{j1}x_{j2}\dots x_{jm}\mu$$

and therefore

$$(x_{11}\dots x_{1m}\mu)\dots(x_{n1}\dots x_{nm}\mu)\nu = (x_{11}\dots x_{n1}\nu)(\varepsilon_1\dots\varepsilon_m\mu) =$$
$$= (x_{11}\dots x_{n1}\nu\varepsilon_1)\dots(x_{11}\dots x_{n1}\nu\varepsilon_m)\mu =$$
$$= ((x_{11}\varepsilon_1)\dots(x_{n1}\varepsilon_1)\nu)\dots((x_{11}\varepsilon_m)\dots(x_{n1}\varepsilon_m)\nu)\mu =$$
$$= (x_{11}\dots x_{n1}\nu)\dots(x_{1m}\dots x_{nm}\nu)\mu.$$

This completes the proof.

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