Bernstein-type inequalities for families of multiplier operators in Banach spaces with Cesàro decompositions. I. General theory

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Dedicated to Professor B. Sz.-Nagy on the occasion of his 60th birthday on July 29, 1973, in admiration

1. Introduction

Let $(\Pi_n)_{2\pi}$ be the set of all trigonometric polynomials $f(x)=\sum_{k=-n}^{n} c_k e^{ikx}$ of degree n. The classical Bernstein inequality states that

$$\|(d/dx)^r f(x)\|_{X_{2\pi}} \leqq n^r \|f(x)\|_{X_{2\pi}} \qquad (f\in(\Pi_n)_{2\pi}), \tag{1.1}$$

where $X_{2\pi}$ is any of the spaces $L^p_{2\pi}$, $1\leqq p<\infty$, or $C_{2\pi}$ of periodic functions (cf. Section 3). As is well known, this inequality plays a central role in the proof of inverse theorems concerning best approximation by trigonometric polynomials. In a very general setting it was recently shown in some basic work of BUTZER—SCHERER [3, 4] (see also [6, 7]) that one may always obtain inverse approximation theorems, provided an inequality of type (1. 1) is available. In their spirit we may formulate the following problem:

Let X be an arbitrary (real or complex) Banach space, $[X]$ the Banach algebra of all bounded linear operators of X into itself, and let $\{T(\varrho)\}_{\varrho>0}\subset[X]$ be a family of operators depending on a parameter $\varrho>0$ (tending to infinity). Suppose B to be a closed linear operator with domain $D(B)\subset X$ and range in X. The family $\{T(\varrho)\}$ is said to satisfy a Bernstein-type inequality (with respect to B) if $T(\varrho)(X)\subset D(B)$ for each $\varrho>0$, and if there exists $\Omega(\varrho)>0$, defined on $(0,\infty)$, and a constant $A>0$ such that

$$\|BT(\varrho)f\| \leqq A\Omega(\varrho)\|f\| \qquad (f\in X,\ \varrho>0). \tag{1.2}$$

In this paper we would like to study (1. 2) in the setting of [2], i.e., the operators in question are generated via multipliers in connection with Fourier expansions corresponding to general decompositions of Banach spaces. Then Bernstein inequalities of type (1. 2) in fact lead to a study of uniformly bounded multipliers (cf. (2. 4)). This is considered in Section 2 which gives convenient sufficient criteria in connection with Cesàro-(C, j)-decompositions. The most concrete version regarding uniform bounds is given in Corollary 2. 4 for multipliers of Fejér's type. This is in fact induced by a fundamental work of SZ.-NAGY [12] on the representation of functions as trigonometric integrals. Indeed, the case $j=1$ of Corollary 2. 4 may be considered as an elementary version of general results in [12] which are in turn used there as multiplier criteria to establish far reaching direct approximation theorems for trigonometric polynomials. Section 3 is concerned with particular choices of $\{T(\varrho)\}$ and B for arbitrary spaces X and decompositions. At the end of this section the trigonometric system is considered, mainly to discuss the question to which extent the classical inequalities may be covered by the present methods. The main bulk of concrete applications, however, will follow in Part II, thus illustrating the usefulness of this simple but nevertheless general and unifying approach to the subject. Finally, let us emphasize that we do not plan to reconstruct the (sometimes) long development of certain instances of Bernstein-type inequalities; for a brief historical account one may consult [10] (seemingly the latest paper on the subject of a survey nature).

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2. Bernstein-type inequalities

Let $\mathbf{Z}$, $\mathbf{P}$, $\mathbf{N}$ be the sets of all, of all non-negative, of all positive integers, respectively. Let $\{P_k\}_{k\in\mathbf{P}}$ be a total sequence of mutually orthogonal continuous projections on X, i.e., (i) $P_k\in[X]$ for each $k\in\mathbf{P}$, (ii) $P_k f=0$ for all $k\in\mathbf{P}$ implies $f=0$, (iii) $P_j P_k=\delta_{jk}P_k$, δ_{jk} being Kronecker's symbol. Then with each $f\in X$ one may associate its unique Fourier series expansion

(2. 1) $$f\sim\sum_{k=0}^{\infty}P_k f \qquad (f\in X).$$

With s the set of all sequences $\gamma=\{\gamma_k\}_{k\in\mathbf{P}}$ of scalars, $\gamma\in s$ is called a multiplier for X (corresponding to $\{P_k\}$) if for each $f\in X$ there exists an element $f^\gamma\in X$ such that $\gamma_k P_k f=P_k f^\gamma$ for all $k\in\mathbf{P}$, thus

(2. 2) $$f^\gamma\sim\sum_{k=0}^{\infty}\gamma_k P_k f \qquad (f\in X).$$

Obviously, $Gf=f^\gamma$ defines a bounded linear operator G on X by the closed graph theorem. Conversely, operators T on X which permit an expansion of type (2. 2), i.e. $P_k(Tf)=\tau_k P_k f$, are called multiplier operators. Denoting the set of all multipliers for X by $M=M(X;\{P_k\})$, with the natural vector operations, coordinate-wise multiplication, and norm

$$\|\gamma\|_M = \sup\{\|f^\gamma\| \,|\, f\in X,\ \|f\|\leqq 1\}, \tag{2.3}$$

M is a commutative Banach algebra, isometrically isomorphic to the subspace $[X]_M\subset[X]$ of multiplier operators on X.

Let $\alpha\in s$ be arbitrary and let X^α be the set of all $f\in X$ for which there exists $f^\alpha\in X$ such that $\alpha_k P_k f=P_k f^\alpha$ for all $k\in\mathbf{P}$. Obviously, if B^α is the operator with domain $X^\alpha\subset X$ and range in X defined by $B^\alpha f=f^\alpha$, then B^α is a closed linear operator for each $\alpha\in s$. Furthermore, if $\{P_k\}$ is fundamental, i.e., the linear span of $\bigcup_{k=0}^{\infty} P_k(X)$ is dense in X, then B^α is densely defined for each $\alpha\in s$.

On restricting oneself to operators with the above multiplier structure one may rephrase problem (1. 2) in terms of the corresponding sequences, namely

Theorem 2. 1. *Let $\alpha\in s$ and $\{T(\varrho)\}\subset[X]_M$ be a family of multiplier operators with associated multipliers $\tau(\varrho)$. If $\alpha\tau(\varrho)\in M$ for each $\varrho>0$, and if there exists $\Omega(\varrho)>0$ and a constant $A>0$ such that*

$$\|\alpha\tau(\varrho)/\Omega(\varrho)\|_M\leqq A \tag{2.4}$$

uniformly for $\varrho>0$, then $\{T(\varrho)\}$ satisfies the Bernstein-type inequality

$$\|B^\alpha T(\varrho)f\|\leqq A\Omega(\varrho)\|f\| \qquad (f\in X,\ \varrho>0). \tag{2.5}$$

Indeed, let $U^{\alpha\tau(\varrho)}\in[X]_M$ be associated with $\alpha\tau(\varrho)$. Then for any $f\in X$, $\varrho>0$, and $k\in\mathbf{P}$

$$P_k(U^{\alpha\tau(\varrho)}f)=\alpha_k\tau_k(\varrho)P_k f=\alpha_k P_k\big(T(\varrho)f\big),$$

so that $T(\varrho)(X)\subset X^\alpha$ and $B^\alpha T(\varrho)f=U^{\alpha\tau(\varrho)}f$. In view of (2. 3—4) this implies (2. 5).

Therefore, in the present setting, the problem is to verify the multiplier condition, particularly (2. 4), thus to establish convenient criteria concerning uniformly bounded multipliers. To this end we follow up the lines of [2] (see also the literature cited there), assuming (essentially) that $\{P_k\}$ is a Cesàro-(C,j)-decomposition of X. For basic facts concerning those decompositions (and bases) one may consult [8], [9], [11].

Let the (C,j)-means of (2. 1) be defined for $j\in\mathbf{P}$ by

$$(C,j)_n f=(A_n^j)^{-1}\sum_{k=0}^{n}A_{n-k}^j P_k f,\qquad A_n^j=\binom{n+j}{n}. \tag{2.6}$$

Obviously $(C,j)_n$ coincides for $j=0$ with the nth partial sum operator $S(n)=\sum_{k=0}^{n} P_k$. For some fixed $j\in\mathbf{P}$ assume that $(C,j)_n$ is uniformly bounded, i.e.

$$\|(C,j)_n f\| \leqq C_j\|f\| \qquad (f\in X), \tag{2.7}$$

the constant $C_j(\geqq 1)$ being independent of $n\in\mathbf{P}$ and $f\in X$.

Remark. In many cases of interest (cf. Part II) one deals with Fourier series in X associated with a total biorthogonal system $\{f_k, f_k^*\}$, $\{f_k\}\subset X$, $\{f_k^*\}\subset X^*$ (the dual of X). Then (2.1) and (2.2) read

$$f\sim \sum_{k=0}^{\infty} f_k^*(f)f_k, \qquad Tf\sim \sum_{k=0}^{\infty}\tau_k f_k^*(f)f_k, \tag{2.8}$$

respectively; $P_k(X)$ is the one-dimensional linear space spanned by f_k. If, furthermore, $\{f_k\}$ is fundamental, then it is clear by the Banach—Steinhaus theorem that (2.7) for $j=0$ is equivalent to the assumption that $\{f_k\}$ is a Schauder basis, i.e., for every $f\in X$

$$\lim_{n\to\infty}\left\|\sum_{k=0}^{n} f_k^*(f)f_k-f\right\|=0,$$

whereas for $j=1$ condition (2.7) is equivalent to the statement that $\{f_k\}$ is a Cesàro basis, i.e., for every $f\in X$

$$\lim_{n\to\infty}\left\|\sum_{k=0}^{n}\left(1-\frac{k}{n+1}\right)f_k^*(f)f_k-f\right\|=0.$$

To study multipliers in connection with systems $\{P_k\}$ satisfying (2.7), let us introduce the following spaces of (scalar-) sequences:

$$bv_{j+1}=\left\{\gamma\in l^\infty \;\middle|\; \|\gamma\|_{bv_{j+1}}=\sum_{k=0}^{\infty}\binom{k+j}{j}|\Delta^{j+1}\gamma_k|+\lim_{m\to\infty}|\gamma_m|<\infty\right\}, \tag{2.9}$$

$$l^\infty=\{\gamma\in s \mid \sup_{k\in\mathbf{P}}|\gamma_k|<\infty\},\quad \Delta\gamma_k=\gamma_k-\gamma_{k+1},\quad \Delta^{j+1}=\Delta^j\Delta.$$

Note that $\gamma\in l^\infty$ and the convergence of the series in (2.9) imply the existence of the limit $\lim_{m\to\infty}\gamma_m=\gamma_\infty$. Furthermore, $bv_{j+1}\subset bv_j$ in the sense of continuous embedding (cf. [5]). Obviously, bv_{j+1} is the space of all sequences of bounded variation if $j=0$, and the space of all bounded, quasi-convex sequences if $j=1$, respectively.

Theorem 2.2. *Let $\{P_k\}\subset[X]$ be a total sequence of mutually orthogonal projections satisfying* (2.7) *for some $j\in\mathbf{P}$. Then every $\gamma\in bv_{j+1}$ is a multiplier and*

$$\|\gamma\|_M\leqq C_j\|\gamma\|_{bv_{j+1}}. \tag{2.10}$$

Indeed, to each $f \in X$ one may associate (cf. [2II])

$$f^{\gamma} = \sum_{k=0}^{\infty} \binom{k+j}{j} \Delta^{j+1} \gamma_k \cdot (C, j)_k f + \gamma_{\infty} f.$$

Therefore, to verify (2. 4) one has to check whether (for suitably chosen $\Omega(\varrho)$) the bv_{j+1}-norms of the sequences $\{\alpha_k \tau_k(\varrho)/\Omega(\varrho)\}_{k \in \mathbf{P}}$ are uniformly bounded for $\varrho > 0$. For this purpose, let BV_{j+1} be the class of all bounded continuous functions f on $[0, \infty)$ for which $f, \dots, f^{(j-1)}$ are locally (i.e. on every compact subinterval) absolutely continuous on $(0, \infty)$ and $f^{(j)}$ is locally of bounded variation on $(0, \infty)$ such that $\int_0^{\infty} x^j |df^{(j)}(x)| < \infty$.

Then one may use the following result (cf. [2II])

Theorem 2. 3. *Let $\gamma \in s$ be such that there exists a function $g \in BV_{j+1}$ with $\gamma_k = g(k)$. Then $\gamma \in bv_{j+1}$ and*

$$\sum_{k=0}^{\infty} \binom{k+j}{j} |\Delta^{j+1} \gamma_k| \leqq D \int_0^{\infty} x^j |dg^{(j)}(x)|, \tag{2. 11}$$

the constant D being independent of γ and j.

As an immediate consequence one has the following criterion concerning uniformly bounded multipliers.

Corollary 2. 4. *Let the system $\{P_k\}$ satisfy (2. 7) for some $j \in \mathbf{P}$. Let $\{\gamma(\varrho)\} \subset s$ be such that there exists $\{g_\varrho\} \subset BV_{j+1}$ with $\lim_{x \to \infty} g_\varrho(x) = 0$ and $\gamma_k(\varrho) = g_\varrho(k)$ for each $k \in \mathbf{P}$, $\varrho > 0$. Then*

$$\|\gamma(\varrho)\|_M \leqq C_j D \int_0^{\infty} x^j |dg_\varrho^{(j)}(x)|. \tag{2. 12}$$

In particular, if g_ϱ is of Fejér's type, i.e., there exists $G \in BV_{j+1}$ such that $g_\varrho(x) = G(x/\varrho)$, then $\{\gamma(\varrho)\}$ is a family of uniformly bounded multipliers.

3. Particular operators in arbitrary spaces

Let X be an arbitrary Banach space and $\{P_k\} \subset [X]$ be any total system of orthogonal projections satisfying (2. 7) for some $j \in \mathbf{P}$. In this section we would like to discuss certain particular choices of families $\{T(\varrho)\}$ and sequences α. Throughout this section A stands for constants which may generally be distinct.

First, let us consider Bernstein inequalities of the classical type (1. 1). Here it is essential that the elements f only belong to the direct sum

$$\Pi_n = \bigoplus_{k=0}^{n} P_k(X) = \left\{ f \in X \,\middle|\, f = \sum_{k=0}^{n} P_k f \equiv S(n) f \right\}$$

rather than to the whole space X. In reducing this situation to that of Theorem 2. 1, we will have to restrict ourselves to the cases $j=0$ or $j=1$.

In case $j=0$ one has $\|\sigma(n)\|_M \leqq C_0$ by hypothesis, $\sigma(n) \in M$ being associated with the partial sum operator $S(n)$. For given non-negative $\alpha \in s$ consider $\alpha\sigma(n)$, the continuous parameter ϱ being replaced by the discrete one n. Since $\alpha\sigma(n) = \beta(n)\sigma(n)$ with $\beta_k(n) = \alpha_k$ for $0 \leqq k \leqq n$, $= \alpha_n$ for $k > n$, Theorems 2. 1—2 imply

$$\|B^\alpha f\| \leqq A\alpha_n \|f\| \qquad (f \in \Pi_n), \tag{3.1}$$

provided $\|\beta(n)\|_{bv_1} \leqq A\alpha_n$ for all $n \in \mathbf{P}$. In particular, if α is monotonely increasing on $\mathbf{P}$, then $\|\beta(n)\|_{bv_1} = \alpha_n - \alpha_0$.

In case $j=1$ consider the family $\{I(n)\} \subset [X]_M$ with associated $\iota(n) \in M$, defined by $\iota_k(n) = 1$ for $0 \leqq k \leqq n$, $= 2-(k/n)$ for $n < k \leqq 2n$, $=0$ for $k > 2n$. Then $\iota(n) \in bv_2$ uniformly for $n \in \mathbf{P}$, and the restriction of $I(n)$ to Π_n is the identity mapping. For given non-negative $\alpha \in s$ consider $\alpha\iota(n)$. Since $\alpha\iota(n) = \eta(n)\iota(n)$ with

$$\eta_k(n) = \alpha_k \text{ for } 0 \leqq k \leqq 2n, = \alpha_{2n} \text{ for } k > 2n, \tag{3.2}$$

it is sufficient to examine $\|\eta(n)\|_{bv_2}$ in order to apply Theorems 2. 1—2. 2. Thus for the restriction of $B^\alpha I(n)$ to Π_n we have

Proposition 3. 1. *Let the system* $\{P_k\}$ *satisfy* (2. 7) *for* $j=1$. *Let* $\alpha \in s$ *be non-negative and assume that* $\eta(n)$ *is defined by* (3. 2) *and satisfies* $\|\eta(n)\|_{bv_2} \leqq A\alpha_{2n}$ *for all* $n \in \mathbf{P}$. *Then*

$$\|B^\alpha f\| \leqq A\alpha_{2n} \|f\| \qquad (f \in \Pi_n). \tag{3.3}$$

In particular, Proposition 3. 1 immediately applies to concave sequences α. For, then α is monotonely increasing so that also $\eta(n)$ of (3. 2) is concave, and thus $\|\eta(n)\|_{bv_2} = \alpha_{2n} - \alpha_0$. Concerning convex sequences α compare the remarks at the end of this section.

In this paper we restrict ourselves to three illustrative examples of sequences α, the significance of this choice in approximation theory being exhibited in [6, 7]. Let $\omega > 0$ be arbitrary, fixed. Then

$$\text{(i) } \alpha = \{k^\omega\}_{k \in \mathbf{P}}, \quad \text{(ii) } \alpha = \{\log(1+k^\omega)\}_{k \in \mathbf{P}}, \quad \text{(iii) } \alpha = \{e^{a(k)}\}_{k \in \mathbf{P}}, \tag{3.4}$$

where $a(x)$ is a non-negative function, defined and monotonely increasing on $[0, \infty)$.

Obviously, (3. 1) applies to (3. 4) (iii) in case $j=0$. Concerning examples (3. 4) (i), (ii) it follows for the corresponding $\eta(n)$ (cf. (3. 2)) that $\|\eta(n)/\alpha_{2n}\|_{bv_2}\leqq A$ uniformly for $n>0$ by Corollary 2. 4 (for (ii) cf. (2. 12)). Thus

Corollary 3. 2. (a) *Let the system* $\{P_k\}$ *satisfy* (2. 7) *for* $j=0$. *Given* $a(x)$ *as specified in* (3. 4) (iii), *then for any* $f\in X$, $n\in\mathbf{P}$

$$\left\|\sum_{k=0}^{n} e^{a(k)}P_kf\right\| \leqq Ae^{a(n)}\left\|\sum_{k=0}^{n} P_kf\right\|. \tag{3. 5}$$

(b) *Let the system* $\{P_k\}$ *satisfy* (2. 7) *for* $j=1$. *Then for any* $\omega>0$ *and* $f\in X$, $n\in\mathbf{P}$

$$\left\|\sum_{k=0}^{n} k^{\omega}P_kf\right\| \leqq An^{\omega}\left\|\sum_{k=0}^{n} P_kf\right\|, \tag{3. 6}$$

$$\left\|\sum_{k=0}^{n} \log(1+k^{\omega})P_kf\right\| \leqq A\log(1+n^{\omega})\left\|\sum_{k=0}^{n} P_kf\right\|. \tag{3. 7}$$

In each case the constant A is independent of $f\in X$, $n\in\mathbf{P}$.

Now, let us apply Theorem 2. 1 directly to several particular families $\{T(\varrho)\}$. We consider the Abel—Cartwright means of order $\varkappa>0$ of the Fourier series (2. 1) of f

$$\text{(i)}\ W_\varkappa(\varrho)f\sim\sum_{k=0}^{\infty} e^{-(k/\varrho)^\varkappa}P_kf \qquad (f\in X,\ \varrho>0), \tag{3. 8}$$

the Bessel potentials of order $\varkappa>0$

$$\text{(ii)}\ L_\varkappa(\varrho)f\sim\sum_{k=0}^{\infty}(1+(k/\varrho)^2)^{-\varkappa/2}P_kf \qquad (f\in X,\ \varrho>0), \tag{3. 8}$$

and the Riesz means of order $\varkappa,\lambda>0$ ($\varrho=n+1\in\mathbf{N}$ being discrete)

$$\text{(iii)}\ R_{\varkappa,\lambda}(n)f\sim\sum_{k=0}^{n}\left(1-\left(\frac{k}{n+1}\right)^{\varkappa}\right)^{\lambda}P_kf \qquad (f\in X,\ n\in\mathbf{P}). \tag{3. 8}$$

Since (cf. [2II]) $\|P_k\|_{[X]}\leqq Ak^j$ in case (2. 7) holds for $j\in\mathbf{P}$, one has equality for all $\varrho>0$ in (i) for $\varkappa>0$, in (ii) for $\varkappa>j+1$, and trivially in (iii) for $\varkappa,\lambda>0$. Furthermore, $L_\varkappa(\varrho)\in[X]_M$ for all $\varkappa>0$ since $(1+x^2)^{-\varkappa/2}\in BV_{j+1}$.

For these families $\{T(\varrho)\}$ let us consider $\alpha=\{k^\omega\}$, $\omega>0$, with $\Omega(\varrho)=\varrho^\omega$. For the corresponding $\alpha\tau(\varrho)$ one has

$$\frac{\alpha_k\tau_k(\varrho)}{\Omega(\varrho)}=g_\varrho(k)=G\left(\frac{k}{\varrho}\right),\quad G(x)=\begin{cases} x^\omega\exp(-x^\varkappa), \\ x^\omega(1+x^2)^{-\varkappa/2}, \\ \begin{cases} x^\omega(1-x^\varkappa)^\lambda, & 0\leqq x\leqq 1, \\ 0\quad, & x>1,\end{cases}\end{cases}$$

respectively. Since $G \in BV_{j+1}$ for each $\omega, \varkappa > 0$ in case (i), for each $0 < \omega < \varkappa$ in case (ii), and for each $\lambda \geqq j$ and $\omega, \varkappa > 0$ in case (iii), it follows by Corollary 2. 4 that

Corollary 3. 3. *Let the system* $\{P_k\}$ *satisfy* (2. 7) *for some* $j \in \mathbf{P}$. *Then for every* $f \in X$, $\varrho > 0$ $(n \in \mathbf{P})$:

$$\left\| \sum_{k=0}^{\infty} k^{\omega} e^{-(k/\varrho)^{\varkappa}} P_k f \right\| \leqq A \varrho^{\omega} \|f\| \qquad (\varkappa, \omega > 0), \tag{3.9}$$

$$\|B^{\{k^{\omega}\}} L_{\varkappa}(\varrho) f\| \leqq A \varrho^{\omega} \|f\| \qquad (0 < \omega < \varkappa), \tag{3.10}$$

where for $0 < \omega < \varkappa - j - 1$ *the corresponding sum exists and therefore*

$$\left\| \sum_{k=0}^{\infty} k^{\omega} (1 + (k/\varrho)^2)^{-\varkappa/2} P_k f \right\| \leqq A \varrho^{\omega} \|f\| \qquad (0 < \omega < \varkappa - j - 1),$$

$$\left\| \sum_{k=0}^{n} k^{\omega} \left(1 - \left(\frac{k}{n+1}\right)^{\varkappa}\right)^{\lambda} P_k f \right\| \leqq A n^{\omega} \|f\| \qquad (\lambda \geqq j; \varkappa, \omega > 0). \tag{3.11}$$

Analogously, Bernstein-type inequalities may be derived for further sequences α.

Remark. The methods employed here may also be used to treat the following counterpart to the general problem (1. 2):

Let $\{T^{(1)}(\varrho)\}$, $\{T^{(2)}(\varrho)\} \subset [X]$ be two families of operators and B a closed linear operator with domain $D(B) \subset X$ and range in X. The family $\{T^{(1)}(\varrho)\}$ is said to satisfy a Bernstein-type inequality (with respect to B and $\{T^{(2)}(\varrho)\}$) if $T^{(1)}(\varrho)(X) \subset \subset D(B)$ for each $\varrho > 0$, and if there exists $\Omega(\varrho) > 0$ such that

$$\|B T^{(1)}(\varrho) f\| \leqq \Omega(\varrho) \|T^{(2)}(\varrho) f\| \qquad (f \in X, \varrho > 0). \tag{3.12}$$

From the point of view of applications following in Part II, however, formulations (1. 2) and (3. 12) are parallel.

Furthermore, note that (2. 5) may be interpreted as a weak and (3. 1), (3. 3) as strong Bernstein-type inequalities, respectively, as introduced in Butzer—Scherer [3, 4]. However, for commutative operators (as considered here), (2. 5) may be sharper than (3. 1), (3. 3), as the particular de la Vallée Poussin process shows (cf. [3]). In the noncommutative case, strong Bernstein-type inequalities seem to be essential.

So far, we have discussed the results of Section 2 in connection with certain particular choices of families $\{T(\varrho)\}$ and sequences α for arbitrary Banach spaces X and systems $\{P_k\}$. Thus it remains to specify X and $\{P_k\}$. However, this will be examined in detail in Part II, devoted to explicit applications to classical orthogonal expansions. Here we only consider the trigonometric system in order to provide a feeling to which extent the classical results are covered by the present approach.

Let $X_{2\pi}=L^p_{2\pi}$, $1\leqq p\leqq\infty$, or $C_{2\pi}$ be the Banach space of 2π-periodic functions with standard norm $\|\cdot\|_{X_{2\pi}}$:

$$\Big(\int_{-\pi}^{\pi}|f(x)|^p dx\Big)^{1/p}(1\leqq p<\infty),\quad \text{ess. sup}\,|f(x)|,\quad \max|f(x)|,$$

respectively. Defining the system $\{P_k\}_{k\in\mathbf{P}}$ by

$$(3.13)\qquad (P_0 f)(x)=f^\wedge(0),\quad (P_k f)(x)=f^\wedge(k)e^{ikx}+f^\wedge(-k)e^{-ikx}\qquad (k\in\mathbf{N}),$$

$f^\wedge(k)$ being the usual Fourier coefficient

$$f^\wedge(k)=(1/2\pi)\int_{-\pi}^{\pi}f(x)e^{-ikx}\,dx\qquad (k\in\mathbf{Z}),$$

$\{P_k\}$ is a total sequence of mutually orthogonal continuous projections on $X_{2\pi}$ and

$$(3.14)\qquad f\sim\sum_{k=0}^{\infty}P_k f\left(\equiv\sum_{k=-\infty}^{\infty}f^\wedge(k)e^{ikx}\right)\qquad (f\in X_{2\pi}).$$

It is well known that $\{P_k\}$ satisfies (2.7) with $j=0$ in case $X_{2\pi}=L^p_{2\pi}$, $1<p<\infty$, and with $j=1$ in all $X_{2\pi}$-spaces. Thus an application of (3.6) yields for any $\omega>0$

$$(3.15)\qquad \Big\|\sum_{k=-n}^{n}|k|^\omega c_k e^{ikx}\Big\|_{X_{2\pi}}\leqq An^\omega\Big\|\sum_{k=-n}^{n}c_k e^{ikx}\Big\|_{X_{2\pi}}.$$

Note that $\sum_{k=-n}^{n}|k|^\omega c_k e^{ikx}$ corresponds to the ωth Riesz derivative $t_n^{\{\omega\}}(x)$ of the trigonometric polynomial $t_n(x)=\sum_{k=-n}^{n}c_k e^{ikx}$ (for the definition and basic properties of this fractional derivative see [1, Sec. 11.5]).

Obviously, apart from the constants, (3.15) coincides with the classical inequality (1.1), thus with

$$(3.16)\qquad \Big\|\sum_{k=-n}^{n}(ik)^r c_k e^{ikx}\Big\|_{X_{2\pi}}\leqq n^r\Big\|\sum_{k=-n}^{n}c_k e^{ikx}\Big\|_{X_{2\pi}}$$

only in case of even values of r. The case of odd values, particularly $r=1$, is not covered for arbitrary $X_{2\pi}$-spaces.

Of course, there are several proofs of (3.16) for $r=1$ and all spaces $X_{2\pi}$, using particular features of the trigonometric system. Here we may mention the classical proof of F. Riesz. In its extended form (cf. [6, 7]) it deals with (even or odd) sequences $\{\alpha_k\}_{k=-\infty}^{\infty}$, non-negative and convex on $\mathbf{P}$ with $\alpha_0=0$. Taking into account addition formulae, specific for the trigonometric system, the proof of the inequality

$$\Big\|\sum_{k=-n}^{n}\alpha_k c_k e^{ikx}\Big\|_{X_{2\pi}}\leqq 2\alpha_n\Big\|\sum_{k=-n}^{n}c_k e^{ikx}\Big\|_{X_{2\pi}}$$

reduces to a verification of the convexity on **P** of the sequence α_{n-k}/α_n for $0 \leqq k \leqq n$, 0 for $k > n$. Whether this method of proof may be extended to more general systems $\{P_k\}$ remains open.

Finally, let us observe that the classical Bernstein inequality (3. 16) for $r=1$, $X_{2\pi}=C_{2\pi}$, for example, may of course be derived by using different methods as a (direct) consequence of theorems in arbitrary Banach spaces. Thus, for example, one may take (3. 16) for $r=2$ and interpolation techniques in order to establish (3. 16) for any $0<r<2$ (see [13]).

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