Principal sequences and stationary sets

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Dedicated to Professor Béla Sz.-Nagy on his 60th birthday

1. Introduction. Assume we are given two regular cardinals ϱ and $\varkappa$ with $\varrho<\varkappa$, and let us assign to each ordinal $\xi<\varkappa$ that is cofinal to ϱ a principal sequence of order type ϱ, i.e. a sequence of ordinals that is strictly monotonic and continuous and tends to ξ. A natural question to ask whether there are many among these sequences which have the same beginning. This type of question was first raised by B. Rotman. The method we use in answering this problem as far as we can is a generalization of that of ROTMAN [4] and FODOR [3].

2. Notations and terminology. Our considerations below can most naturally be carried out in the framework of Zermelo—Fraenkel set theory with the Axiom of Choice. We use the usual set-theoretical conventions and notations. Thus we consider an ordinal as the set of its predecessors, and a cardinal is identified with its initial ordinal. Ordinals are usually denoted by lower case Greek letters, and the sign + denotes ordinal addition. dom (f) denotes the domain, and $f \restriction x$ the restriction to the set x of the function f. Braces $\{,\}$ are used to define sets, and angle brackets $\langle,\rangle$ to define functions; that is, if f is a function, then by definition we have

$$f=\langle f(x):\ x\in \mathrm{dom}(f)\rangle.$$

A *sequence* is a function whose domain is an ordinal. The word monotone is meant in the wider sense; if we mean strictly monotone, then we say so. A monotone sequence $\langle \xi_\alpha : \alpha<\eta\rangle$ of ordinals tends to ξ if $\xi=\bigcup_{\alpha<\eta}\xi_\alpha$. A principal sequence is a strictly monotone sequence of ordinals that is continuous; here a monotone sequence $\langle \xi_\alpha : \alpha<\eta\rangle$ of ordinals is called continuous if $\xi_\alpha=\bigcup_{\beta<\alpha}\xi_{\beta+1}$ holds for any $\alpha<\eta$.

Given a regular cardinal $\varkappa>\omega$, a set $X\subseteq\varkappa$ is called *stationary* (in $\varkappa$) if it meets every *closed unbounded* subset of $\varkappa$. (Here the topology on $\varkappa$ is that generated by its natural ordering; unbounded means cofinal to $\varkappa$.) A key fact proved by the author is that the nonstationary subsets of $\varkappa$ form a normal ideal on $\varkappa$; that is, if a function f sending ordinals to ordinals is called *regressive* whenever $f(\alpha)<\alpha$ holds for all nonzero α in its domain, then we have (see [2, Satz 2 on p. 141]):

2. 1. Theorem. *Any regressive function the domain of which is a stationary set (in some fixed regular cardinal) is constant on a stationary set.*

3. The next lemma, which is a generalization of results in [3] and [4], will be useful in dealing with our main problem outlined in the introduction. The lemma may be interesting in itself, and it has other applications as well (it enables one to derive special cases of Solovay's decomposition theorem [5] which says that every set stationary in a regular cardinal $\varkappa>\omega$ can be split into $\varkappa$ stationary sets (cf. [3]).

3. 1. Lemma. *Assume ϱ and $\varkappa$ are two infinite regular cardinals, $\varrho<\varkappa$, and S is a set stationary in $\varkappa$ of ordinals less than $\varkappa$ and cofinal to ϱ. Assume, further, that for every $\xi\in S$ $\langle f_\alpha(\xi):\alpha<\varrho\rangle$ is a monotone sequence tending to ξ. Then there exists an ordinal $\nu_0<\varrho$ such that, for every ν with $\nu_0\leqq\nu<\varrho$ there is a set $F(\nu)\subseteqq\varkappa$ of cardinality $\varkappa$ such that the set*

$$\{\xi\in S: f_\nu(\xi)=\gamma\}$$

is a stationary for each $\gamma\in F(\nu)$.

Proof. Call a function f mapping a subset of $\varkappa$ into $\varkappa$ essentially bounded if there exists an ordinal $\alpha<\varkappa$ such that

$$\{\xi\in\mathrm{dom}(f): f(\xi)<\alpha\}$$

is nonstationary. We assert that at least one of the functions f_ν is not essentially bounded. In fact, assuming that α_ν is an essential bound for f_ν, we see that $\alpha=\bigcup_{\nu<\varrho}\alpha_\nu$ is a common essential bound for each f_ν; moreover, it is clear that the set

$$H=\{\xi\in S:\exists\nu<\varrho[f_\nu(\xi)>\alpha]\}\,(\subseteqq\bigcup_{\nu<\varrho}\{\xi\in S: f_\nu(\xi)>\alpha_\nu\})$$

is nonstationary. On the other hand $H=S-\alpha$, as by our assumptions we have $\xi=\bigcup_{\nu<\varrho}f_\nu(\xi)$ for every $\xi\in S$. This contradiction implies that there is indeed a $\nu_0<\varrho$ such that f_{ν_0} is not essentially bounded.

Now take an arbitrary ν with $\nu_0\leqq\nu<\varrho$. Obviously, f_ν is not essentially bounded, as $f_{\nu_0}(\xi)\leqq f_\nu(\xi)$ holds for every $\xi\in S$. Put $F(\nu)=\{\gamma<\varkappa:\{\xi\in S: f_\nu(\xi)=\gamma\}$ is stationary$\}$. We are going to show that $F(\nu)$ is cofinal to $\varkappa$, and so it has cardinality $\varkappa$. Indeed, take any $\alpha<\varkappa$. As f_ν is not essentially bounded, the set

$$X=\{\xi\in S: f_\nu(\xi)\geqq\alpha\}$$

is stationary. So, applying Theorem 2. 1 to the function $f\restriction X$, we see that there is a $\gamma\geqq\alpha$ such that the set

$$\{\xi\in S: f_\nu(\xi)=\gamma\}$$

is stationary. This means that $\gamma \in F(\nu)$, showing that $F(\nu)$ is cofinal to $\varkappa$, as asserted above.

This completes the proof.

4. Now we are in position to establish the main result of these notes:

4. 1. Theorem. *Assume ϱ and $\varkappa$ are two infinite cardinals such that $\varrho<\varkappa$ and $\tau^\sigma<\varkappa$ holds for any two cardinals σ and τ with $\sigma<\varrho$, $\tau<\varkappa$. Let S be a set stationary in $\varkappa$ of ordinals less than $\varkappa$ and cofinal to ϱ. For every $\xi\in S$, let $\langle f_\nu(\xi):\nu<\varrho\rangle$ be a principal sequence tending to ξ. Then there exists an ordinal $\nu_0<\varrho$ such that for every ν with $\nu_0\leqq\nu<\varrho$ there is a set $G(\nu)$ of principal sequences of type ν of ordinals such that the set $\{\xi\in S:\langle f_\mu(\xi):\mu<\nu\rangle=s\}$ is stationary in $\varkappa$ for each $s\in G(\nu)$.*

Proof. Observe that it is enough to prove this theorem for any ν of form $\eta+1$ such that $\nu_0<\nu<\varrho$, for some $\nu_0<\varrho$. In fact if η is a limit ordinal and $G(\eta+1)$ has already been defined in a way complying with the requirements of the theorem, then we can take

$$G(\eta)=\{s\restriction\eta:\ s\in G(\eta+1)\}.$$

One should only note here that the cardinality of $G(\eta)$ is $\varkappa$, for if $s_1, s_2\in G(\eta+1)$ and $s_1\restriction\eta=s_2\restriction\eta$ then also $s_1=s_2$ by the continuity of these sequences.

Now take ν_0 to be that of the preceding lemma, and let η be an ordinal with $\nu_0\leqq\eta<\varrho$. Select an arbitrary $\gamma\in F(\eta)$, this latter set having been defined in the preceding lemma. In view of the assumption that $\tau^\sigma<\varkappa$ holds for any two cardinals $\tau<\varkappa$ and $\sigma<\varrho$ we can see that there are less than $\varkappa$ different ones among the sequences

$$\langle f_\mu(\xi):\ \mu\leqq\eta\rangle$$

as ξ runs over the elements of the stationary set

$$\{\xi\in S:\ f_\eta(\xi)=\gamma\}.$$

So there is a sequence s_γ such that the set

$$\{\xi\in S:\langle f_\mu(\xi):\mu\leqq\eta\rangle=s_\gamma\}$$

is stationary. Set

$$G(\eta+1)=\{s_\gamma:\ \gamma\in F(\eta)\}.$$

The proof is complete.

4. 2. It is not difficult to see that the assumption that $\tau^\sigma<\varkappa$ holds for any two cardinals τ and ϱ with $\tau<\varkappa$, $\sigma<\varrho$ is essential. More exactly, we have

Theorem. *Assume ϱ and $\varkappa$ are two infinite regular cardinals, and $\tau<\varkappa$ and $\sigma<\varrho$ are cardinals such that $\tau^\sigma\geqq\varkappa$. Let S be the set of all ordinals ξ cofinal to ϱ for which $\tau+\sigma\leqq\xi<\varkappa$. Then for every $\xi\in S$, there exists a principal sequences s_ξ tending* *to ξ such that for any sequence s of type $\geqq\sigma$ of ordinals there is at most one $\xi\in S$ for which s is an initial segment of s_ξ.*

Proof. Clearly, one can take $\varkappa$ different principal sequences of type $\sigma+1$ of ordinals $\leqq\tau$; let $\langle t_\xi:\xi\in S\rangle$ be an enumeration of these sequences. For any $\xi\in S$ continue the sequence t_ξ to a principal sequence s_ξ of type ϱ that tends to ξ. The proof is complete.

5. If s_1 and s_2 are two principal sequences of type ϱ, where ϱ is a cardinal, then s_1 and s_2 (or, rather, their ranges) have less than ϱ elements in common. It would, however, be interesting to know the answer to the following

Problem. *Assume ϱ and $\varkappa$ are regular cardinals, $\varrho<\varkappa$. Let S be the set of all ordinals $<\varkappa$ that are cofinal to ϱ. Is then there a principal sequence s_ξ for each $\xi\in S$ such that, for some cardinal $\lambda<\varrho$, the sequences s_η and s_ξ have less than λ elements in common whenever ξ and η are different elements of S.*

By Theorem 4. 1, the answer is in the negative if $\tau^\sigma<\varkappa$ holds for any cardinals $\sigma<\varrho$ and $\tau<\varkappa$. We conjecture, however, that the answer is always in the negative, and remains to be so even if we take S to be any stationary set of ordinals $<\varkappa$ that are cofinal to ϱ.

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(Received September 12, 1972)