Translation invariant transformations of integration spaces

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Dedicated to Professor Béla Szőkefalvi-Nagy on his sixtieth birthday

1. Introduction

A number of writers (e.g. [1—5]) have dealt with the existence and properties of linear transformations between function spaces obeying various functional equations. In almost all cases these equations are of a type that I have termed "appropriate" in my article [1]; this term will now be defined.

Let $A(X)$ be a space of functions defined on a set X. A linear transformation W on $A(X)$ to itself is called appropriate if for each x in X and f in $A(X)$ the value of $Wf(x)$ depends exclusively on the value of f at some point in X, say Vx, or (in the case of spaces of functions defined only up to sets of measure zero) if a similar statement is true in the limit for functionals on $A(X)$ whose support tends to x. Equivalently, W is an appropriate transformation if $Wf(x)=Q(x)f(Vx)$. An appropriate group is a representation of a group by a group of appropriate transformations.

A linear operator T from a space $A(X)$ to a space $B(U)$ of functions on X and U respectively is said to obey an appropriate functional equation if it is an intertwining operator between appropriate groups of transformations on $A(X)$ and $B(U)$: that is to say, if there is a group G represented by appropriate groups $W(g)$ and $W^*(g)$ of transformations on $A(X)$ and $B(U)$ and if T obeys

$$TW(g) = W^*(g)T, \qquad g\in G. \tag{1.1}$$

In [1] I have shown that if X and U are intervals of the real line, and G is the additive group of reals, then after possible splitting of X and U into intervals invariant in the groups $V(g)$ and $V^*(g)$ and changes of variable in X and U all ap-

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propriate functional equations can be reduced to four canonical forms, of which the most important are:

$$\text{I. a} \qquad T\left\{\frac{p(x+h)}{p(x)} f(x+h)\right\}(u) = \frac{q(u+h)}{q(u)} \{Tf(x)\}(u+h),\ u,\ h\in R,$$

$$\text{II. a} \qquad T\left\{\frac{p(x+h)}{p(x)} f(x+h)\right\}(u) = e^{hq(u)} \{Tf(x)\}(u),\ u\in E,\ h\in R$$

for some set $E\subset R$.

If we replace $A(X)$ by $pA(X)$ and $B(U)$ by $qB(U)$ then I.a becomes

$$\text{I.} \qquad [Tf(x+h)](u) = [Tf(x)](u+h),\ u,\ h\in R$$

and if we make only the first of these transformations I.a. becomes

$$\text{II.} \qquad [Tf(x+h)](u) = e^{hq(u)} (Tf(x)](u),\ u\in E,\ h\in R.$$

If X and U are sets in R^k, $k>1$, and G is the group of translations of R^k, then the situation is considerably more complicated and there can be many more essentially distinct forms of appropriate functional equations. The problem of classifying these may be of interest; but it is complicated by the fact that there can be, for example, periodic representations of the translation group. It remains of interest to study equations I and II; I in particular has received considerable attention, notably in the work of HÖRMANDER [5], in which the existence of solutions mapping a space L_p onto a space L_q with respect to Lebesgue measure on R^k is studied.

The reduction of the equation Ia to the form I for transforms over integration spaces involves a change in the measure on the space; and it is therefore of interest to investigate the equation for spaces $L_p(\mu)$ and $L_q(\nu)$ with general Radon measures.

Writing $\tau(h)f(x) = f(x-h)$, the equation I becomes $T\tau(h)=\tau(h)T$. In general there is a difficulty in interpreting this equation; for the translation operator $\tau(h)$ does not necessarily map $L_p(\mu)$ to itself, and if it does not the meaning of the equation is unclear. We therefore start our investigation by studying what properties on μ ensure that $\tau(h)$ is always defined, and also some other properties of $\tau(h)$ that simplify the structure of μ. The following sections then give conditions that are necessary for the existence of nonzero T satisfying the equation I, and also discuss some properties of the solutions.

2. Conditions for existence of $\tau(h)$

Notation. If μ is a positive Radon measure on R^k, $\mathscr{L}_p(\mu)$ is the space of all μ measurable functions $f(x)$ such that $|f(x)|^p$ is μ summable, and $L_p(\mu)$ is the corresponding Banach space of functions modulo null functions. We write m for Lebesgue measure, $m(dx)=dx$, and if $\mu=\lambda m$, that is $\mu(dx)=\lambda(x)dx$, we write $L(p, \lambda)$ for $L_p(\mu)$.

Theorem 1. *Let μ be a positive Radon measure in R^k. The following conditions on μ, for any p, $1 \leqq p < \infty$, are equivalent:*

(a) *If $f \in \mathscr{L}_p(\mu)$ then $\tau(h)f \in \mathscr{L}_p(\mu)$ for all $h \in R^k$;*

(b) *if f and g are in the same equivalence class in $\mathscr{L}_p(\mu)$ so are $\tau(h)f$ and $\tau(h)g$ for any $h \in R^k$;*

(c) *for any h, $\tau(h)$ takes $L_p(\mu)$ into itself;*

(d) *$\tau(h)$ is a continuous map of $L_p(\mu)$ to itself for any h;*

(e) *there is a positive Lebesgue measurable function $\lambda(x)$, bounded with $\lambda(x)^{-1}$ over any compact set of values of x, such that $\lambda(x)dx=\mu(dx)$ and $\|\tau(h)\|_p^p = \sup \dfrac{\lambda(x+h)}{\lambda(x)}$ is bounded over any compact set of values of h.*

(b) and (c) are clearly equivalent, and imply (a). Now let f and g be equivalent in $\mathscr{L}_p(\mu)$ and let $r(x)=0$ if $f(x)=g(x)$, $r(x)=\infty$ otherwise; then r is a μ null function, and so in $\mathscr{L}_p(\mu)$. If $\tau(h)f(x) \neq \tau(h)g(x)$, then $\tau(h)r(x)=\infty$; thus if (a) holds, the set with $\tau(h)r(x)=\infty$ is a null set an so $\tau(h)f(x)=\tau(h)g(x)$ almost everywhere: thus (a) implies (b) and (c).

Let us now write $\tau(h)\mu=\mu_h$, that is

$$\int f(x)\,\mu_h(dx) = \int f(x+h)\,\mu(dx).$$

Our arguments show that (a), (b) or (c) imply that μ is quasi-invariant [7]: μ null sets translate into μ null sets and μ_h is absolutely continuous with respect to μ, so that there is for each h a function $\varphi(x, h)$ nonnegative and μ summable over any set of finite μ measure as a function of x such that $\mu_h(dx)=\varphi(x, h)\mu(dx)$; since μ is absolutely continuous with respect to μ_h it follows that $1/\varphi(x, h)$ is also summable over any set with finite μ_h and so finite μ measure. For any f and h $\int |f(x)|^p \mu(dx)$ is finite if and only if $\int |f(x)|^p \varphi(x, h)\mu(dx)$ is finite. Let $\varphi_n(x, h)=\min(\varphi(x, h), n)$; then the map $f(x) \to \varphi_n(x, h)f(x)$ is bounded on $L_p(\mu)$ to itself for any fixed n and h and the set $\{\varphi_n(x, h)f(x);\ n=1, 2, \dots\}$ is bounded in $L_p(\mu)$ for each f; by the Banach—Steinhaus Theorem it is uniformly bounded, and so $\varphi(x, h)$ is bounded for each h. This proves that $\tau(h)$ is a bounded transformation of $L_p(\mu)$ to itself; if $K(h)= \operatorname{ess\,sup}\{\varphi(x, h): x \in R^k\}$ then $\|\tau(h)\| = K(h)^{1/p}$, $\log \|\tau(h)\| = p^{-1} \log K(h) = p^{-1} L(h)$

say, and since $\tau(h)\tau(h') = \tau(h+h')$ it follows that $L(h)$ is everywhere finite, measurable and subadditive. We now show that $L(h)$ is bounded over any compact set of h. If it is not bounded above, then there is a convergent sequence (h_n) for which $L(h_n) \to \infty$; since, for any h, $L(h+h_n) \geqq L(h_n) - L(-h)$, we can find such a sequence convergent to any assigned h; and we suppose that h has each coordinate h^j positive. For any such h let $C(h) = \{x: 0 \leqq x^j \leqq h^j,\ j = 1, 2, \ldots, k\}$. Since $L(h_n) \leqq L(x) + L(h_n - x)$, either $L(x) > \frac{1}{2}L(h_n)$ or $L(h-x) \geqq \frac{1}{2}L(h_n)$ holds for any given x; and so $m\{x: L(x) \geqq L(h_n);\ x \in C(h_n)\} \geqq \frac{1}{2}mC(h_n)$. We can choose the h_n so that $h_n^j \sim 0$ for all j and n, and $L(h_n) > n$; then the set $\{x: L(x) > \frac{1}{2}n,\ x \in C(h_n)\}$ has for each n Lebesgue measure greater than $\frac{1}{2}mC(h_n) \to \frac{1}{2}mC(h)$ and hence $m\{x: L(x) = \infty,\ x \in C(h)\} \geqq \frac{1}{2} \geqq mC(h)$; but this set is empty, so we have a contradiction. If L is not bounded below on a set C, it is not bounded above on $-C$; hence L is bounded on every compact.

Now by the Lebesgue—Vitali Theorem (e.g. [6], Vol. I, Theorem III. 12. 6)

$$\lim_{r \to 0} \frac{B(a, r)}{mB(a, r)} = \lambda(a) \text{ say}$$

exists for almost all a and is finite. Choose a in $B(0, r)$ so that this holds. Any set E in $B(0, r)$ can be covered by a finite number of translates of a ball $B = B(a, r)$ with $\mu(B) < (\lambda(a)+1)m(B)$, and hence $\mu(E) < M(\lambda(a)+1)m(E)$, where $\log M$ is the upper bound of $L(h)$ in $B(0, 2r)$. Thus μ is absolutely continuous with respect to Lebesgue measure $m: \mu(dx) = \lambda(x)\,dx$ with λ bounded over any compact. Since then $\mu_h(dx) = \lambda(x+h)\,dx$, $\varphi(x, h) = \dfrac{\lambda(x+h)}{\lambda(x)}$ $\|\tau(h)\|^p = \sup \dfrac{\lambda(x+h)}{\lambda(x)}$ is bounded over every compact, and so is $\|\tau(h)\|^{-1}$.

This proves that (a) implies (d) and (e). On the other hand, it is easy to see that (d) or (e) imply (a).

It is essential for the truth of this theorem that $p < \infty$. If $p = \infty$, then for (a) to hold μ must be quasi-invariant, and if this is the case then $\|\tau(h)f\| = \|f\|$ for any f and h. We can conclude again that $\mu(dx) = \lambda(x)\,dx$ with λ locally summable, but not that λ is necessarily bounded over a compact or restricted in growth.

In future we write $L(p, \lambda)$ for $L_p(\mu)$ when $\mu(dx) = \lambda(x)\,dx$. We define $l(x) = l(\lambda, x) = \log \lambda(x)$, $L(h) = L(\lambda, h) = \operatorname{ess\,sup}\,[l(x+h) - l(x)]$.

Theorem 2. *Let μ obey the conditions of Theorem 1 for some p, $1 \leqq p < \infty$. Then $L(\lambda, h)$ is subadditive and for any h*

$$F(\lambda, h) = \lim_{\alpha \to \infty} \frac{L(\lambda, \alpha h)}{\alpha} \tag{2.1}$$

exists. $F(h) = F(\lambda, h)$ is a continuous convex positive homogeneous function, everywhere finite: $F(\lambda, h) \geqq -F(\lambda, -h)$.

If $\|\tau(h)\|$ is continuous in h, then $\lambda(x)$ is continuous and $l(x)$ and $L(h)$ are uniformly continuous, and $L(h)$ is continuous. The limit in (2. 1) *exists uniformly in h over the sphere $\|h\|=1$.*

Theorem 1 shows that $L(h)$ exists and is finite everywhere; it is obviously subadditive. Consequently

$$\frac{L((\alpha+\beta)h)}{\alpha+\beta} \leqq \frac{\alpha}{\alpha+\beta}\frac{L(\alpha h)}{\alpha}+\frac{\beta}{\alpha+\beta}\frac{L(\beta h)}{\beta}$$

for any α, β and so $L(\alpha h)/\alpha$ decreases with α for positive α, and this proves that $F(h)$ exists and is less than ∞. Since $L(\alpha h)+L(-\alpha h) \geqq L(0) = 0$, it follows that $F(h)+F(-h) \geqq 0$ and hence $F(h) \geqq -F(-h) \geqq -\infty$. Clearly $F(\beta h)=\lim L(\alpha\beta h)/\alpha= =\beta F(h)$ for any positive β and $F(h+k) \leqq \lim L(\alpha h)/\alpha+\lim L(\beta h)/\beta = F(h)+F(k)$. It follows that F is positive homogeneous and subadditive, hence that it is convex and so continuous.

Since $p \log \lambda(h) = \text{ess sup}\,\{l(x+h)-l(x)\}$ continuity of $\lambda(h)$ implies that the righthand side tends to 0 as $h\to 0$, that is

(2. 2) $$\text{ess sup}\,[\lambda(x+h)/\lambda(x)] \to l \text{ as } h\to 0$$

$\lambda(x)$ is everywhere equal to $\lim\limits_{r\to 0} \mu B(x, r)/mB(x, r)$ that is, to

$$\lim \int_{\|x-y\|<r} \lambda(y)\, dy/mB(x, r).$$

The corresponding integral for $\lambda(x+h)$ has y replaced with $y+h$ and so since $\lambda(x)$ is bounded over any compact and because of (2. 2), λ is continuous. $l(x)$ is then also continuous and so $\sup\,[l(x+h)-l(x)] = p \log \|\tau(h)\| \to 0$ as $h\to 0$, that is, $l(x)$ is uniformly continuous. Uniform continuity of $L(h)$ follows immediately: if $\delta>0$ is such that $l(x+h)-l(x) < \varepsilon$ when $\|h\|<\delta$ then $|L(h)-L(h')| < \varepsilon$ if $|h-h'| < \delta$.

If the limit in (2. 1) does not exist uniformly over $\|h\|=1$, then for some $\varepsilon>0$ we can select a sequence (h_n), convergent to a point h on the sphere, and indeed such that $\|n(h-h_n)\| \to 0$ as $n\to\infty$, so that for all n,

$$L(nh_n) > n[F(h_n)+3\varepsilon].$$

For each n we can choose x_n such that $l(x_n+nh_n)-l(x_n) > L(nh_n)-\varepsilon$, and then

$$n[F(h_n)+3\varepsilon] < l(x_n+nh_n)-l(x_n) = l(x_n+nh_n)-l(x_n+nh)+l(x_n+nh)-l(x) \leqq$$
$$\leqq L(nh_n-nh)+L(nh) < L(nh_n-nh)+n[F(h)+\varepsilon].$$

For some n_0, $L(nh_n-nh) < \varepsilon$ and $|F(h_n)-F(h)| < \varepsilon$ if $n>n_0$, and then the lefthand side is greater than $n(F(h)+2\varepsilon)$ and the righthand side is less than

$\varepsilon+n[F(h)+\varepsilon]$, so that the inequality cannot hold for all n. This proves the uniform convergence in (2. 1).

Theorem 3. *Let $F(x)$ be the positive homogeneous convex function defined by* (2. 1). *Then*

(a) *for any h and any $\varepsilon>0$ there is an α_0 so that if $\alpha>\alpha_0$*

$$\exp\left[-\alpha\big(F(-h)+\varepsilon\big)\right] < \lambda(\alpha h) < \exp\left\{\alpha\big(F(h)+\varepsilon\big)\right\};$$

(b) *if $\|\tau(h)\|$ is continuous in h then this holds uniformly over $\|h\|=1$, that is, there is an α_0 so that if $\|x\|>\alpha_0$*

$$\exp\left[-\big(F(x)+\varepsilon\|x\|\big)\right] < \lambda(x) < \exp\big(F(x)+\varepsilon\|x\|\big).$$

By definition, $-L(-\alpha h) < l(\alpha h)-l(0) < L(\alpha h)$ for all positive α and all h, so that for large enough α

$$-\alpha[F(-h)+\varepsilon] < l(\alpha h)-l(0) < \alpha[F(h)+\varepsilon]$$

and the first statement follows; the second is a consequence of this and of the uniform convergence of $L(\alpha h)$ to $F(h)$ over the unit sphere.

The problem of the existence of intertwining operators between representations of $\tau(h)$ on integration spaces is clearly linked with the topological properties of the group $\tau(h)$ acting on these spaces: and more precise results can be found if the behaviour of $\tau(h)$ is more closely specified.

For any $R>0$ let us write $L(p, \lambda, R)$ for the set of functions in $L(p, \lambda)$ whose supports are in $B(0, R)$. We examine conditions under which the action of $\tau(h)$ is closely approximated by its action on $L(p, \lambda, R)$.

If, for each $\varepsilon>0$, there is an R such that, for any h

$$\sup\left\{\frac{\|\tau(h)f\|}{\|\tau(h)\|\,\|f\|} : f\in L(p, \lambda, R)\right\} \geqq (1-\varepsilon), \tag{2. 3}$$

then we say that τ is compactly approximated: and if there is an R so that

$$\sup\left\{\frac{\|\tau f\|}{\|\tau(-h)f\|\,\|\tau(h)\|} : f\in L(p, \lambda, R)\right\} \geqq (1-\varepsilon) \tag{2. 4}$$

then τ is inversely compactly approximated.

Theorem 4. *If τ is compactly approximated then for any $\varepsilon>0$ there is for any h an $\alpha_0(h, \varepsilon)$ and a constant A so that if $\alpha>\alpha_0$*

$$Ae^{F(\alpha h)} \leqq \lambda(\alpha h) \leqq e^{F(\alpha h+\varepsilon\alpha)},$$

and if $\|\tau(h)\|$ *is continuous in h, there is an* $\alpha_0(\varepsilon)$ *so that if* $\|x\| > \alpha_0$

$$Ae^{F(x)} \leqq \lambda(x) \leqq e^{F(x)+\varepsilon\|x\|}.$$

If τ *is inversely compactly approximated, then these equations become*

$$e^{-(F(-\alpha h)+\varepsilon\alpha)} \leqq \chi(ah) \leqq Ae^{-F(-\alpha h)},$$

$$e^{-(F(-x)+\varepsilon\|x\|)} \leqq \lambda(x) Ae^{-F(-x)},$$

respectively.

It is easy to see that $\sup\left\{\frac{\|\tau(h)f\|}{\|f\|} : f\in L(p, \lambda, R)\right\}$ is equal to

$$\sup \{[\lambda(x+h)/\lambda(x)]^{1/p};\ \|x\| \leqq R\}$$

and so that τ is compactly approximated if and only if there is for each $\varepsilon > 0$ an R such that, for all h

$$L(h) \geqq \sup \{l(x+h)-l(x): \|x\| < R\} \geqq L(h)-\varepsilon.$$

In that case one has on the one hand that $l(\alpha h)-l(0) \leqq L(\alpha h)$ and on the other that

$$l(\alpha h) = l(\alpha h)-l(x+\alpha h)+l(x)+l(x+\alpha h)-l(x) \leqq$$

$$\leqq -L(-x)+l(x)+l(x+\alpha h)-l(x) \leqq \sup \{l(x)-L(-x): \|x\| < R\}+L(\alpha h)+\varepsilon =$$

$$= A'+L(\alpha h)-\varepsilon,$$

say. Now for $\alpha > \alpha_0(h, \varepsilon)$, $\alpha(F(h)+\varepsilon) \geqq L(\alpha h) \geqq \alpha F(h)$ so that $F(h)+A'/\alpha \leqq \leqq l(\alpha h)/\alpha \leqq l(0)/\alpha + F(h)+\varepsilon$, and so

$$Ae^{F(\alpha h)} \leqq \lambda(\alpha h) \leqq e^{F(\alpha h)+\varepsilon\alpha}.$$

The second inequalities follow from the uniform convergence of $L(\alpha h)/\alpha$ if $\|\tau(h)\|$ is continuous.

The last two inequalities follow by a similar argument, based on the observation that

$$\sup\left\{\frac{\|\tau f\|}{\|\tau(-h) f\|} : f\in L(p, \lambda, R)\right\} = \sup \{[\lambda(x)/\lambda(x-h)]^{1/p} : \|x\| < R\}.$$

The importance of these results to our later arguments is that they give conditions under which the growth of $\lambda(x)$ as $\|x\| \to \infty$ is regular. The approximations to $\lambda(x)$ in these formulae give examples of the E functions defined in the following.

Definition. $E(\lambda, h)$ is an upper E function for λ if $\lambda(x+h)/\lambda(x)E(\lambda, h)$ is bounded for all x and h and if for all x

$$\limsup \frac{\lambda(x+h)}{\lambda(x)E(\lambda, h)} \leqq 1;$$

as h tends to infinity along any ray;

$E(\lambda, h)$ is a lower E function for λ if

$$\liminf \frac{\lambda(x+h)}{\lambda(x)E(\lambda, h)} \geqq 1$$

as h tends to infinity along any ray.

Theorem 5. *If $E(\lambda, h)$ is an upper E function for λ, then for any $f \neq 0$*

$$\limsup \frac{\|\tau(h)f\|}{\|f\| E(\lambda, h)^{1/p}} \leqq 1, \qquad \limsup \frac{\|f+\tau(h)f\|}{\|f\|(1+E(\lambda, h))^{1/p}} \leqq 1$$

$$\limsup \frac{\|\tau(h)f+\tau(-h)f\|}{\|f\|(E(\lambda, h)+E(\lambda, -h)^{1/p})} \geqq 1,$$

as h tends to infinity along any ray. If $E(\lambda, h)$ is a lower E function then the same inequalities hold with signs reversed and lim sup *replaced by* lim inf.

For any h we have

$$\frac{\|\tau(rh)f\|^p}{E(\lambda, rh)} = \int |f(x)|^p \frac{\lambda(x+rh)}{\lambda(x)E(\lambda, rh)} \lambda(x)\,dx \leqq \int |f(x)|^p S(x, h, R)\,dx$$

when $r > R$, if $S(x, h, R) = \sup \{\lambda(x+rh)/\lambda(x)E(\lambda, rh) : r > R\}$. Since this is bounded and has a limit not greater than 1 as $R \to \infty$,

$$\limsup_{r\to\infty} \frac{\|\tau(rh)f\|^p}{E(\lambda, rh)} \leqq \|f\|^p$$

and this proves the first statement. To prove the second and third, we note that for any $\varepsilon > 0$ we can find f_1 such that $f_2 = f - f_1$ has norm less than ε and f_1 has compact support. Then if h is large enough the supports of f_1, $\tau(h)f_1$ and $\tau(-h)f_1$ are disjoint, so that

$$\|f_1 + \tau(h)f_1\|^p = \|f_1\|^p + \|\tau(h)f_1\|^p,$$

$$\|\tau(h)f_1 + \tau(-h)f_1\|^p = \|\tau(h)f_1\|^p + \|\tau(-h)f_1\|^p.$$

Thus, for h sufficiently large in any direction

$$\|f+\tau(h)f\| \leqq \|f_1+\tau(h)f_1\|+\|f_2\|+\|\tau(h)f_2\| \leqq$$
$$\leqq \|f\|\big(1+E(\lambda,h)\big)^{1/p}(1+\varepsilon)+\varepsilon+KE(\lambda,h)^{1/p}\varepsilon$$
$$\leqq \|f\|\big(1+E(\lambda,h)\big)^{1/p}(1+\varepsilon)+K\varepsilon\big(1+E(\lambda,h)\big)^{1/p},$$

where $K=\sup\{\lambda(x+h)/E(\lambda,h)\lambda(x)\}$, and this leads to the second inequality. The third inequality follows in the same way.

The inequalities for a lower E function follow similarly; but the proof relies on Fatou's lemma, and does not need boundedness.

3. Existence of continuous translation invariant operators

We now consider conditions on $\lambda(x)$ and $\mu(x)$ that are necessary for there to be a nonzero continuous linear T mapping $L(p,\lambda)$ to $L(q,\mu)$ and obeying

(3.1) $$[Tf(x+h)](u)=[Tf(x)](u+h),\ u,h\in R.$$

Our first results depend on the following general theorem, which includes many of the special criteria that have been used in such problems.

Theorem 6. *Let S be a directed set and for all s in S let $V(s)$ and $W(s)$ be bounded linear operators on normed spaces A and B respectively, mapping each space to itself, and let $v(s)$ and $w(s)$ be positive valued functions such that, for all $f\in A$ and $g\in B$*

(3.2) $$\limsup_{s}\frac{\|V(s)f\|_A}{v(S)\|f\|_A}\leqq 1,\qquad \liminf_{s}\frac{\|W(s)g\|_B}{w(s)\|g\|_B}\geqq 1.$$

Then if there is a continuous nonzero linear $T\colon A\to B$ such that $TV(s)=W(s)T$ for all $s\in S$ we must have

(3.3) $$\liminf_{s}\frac{v(s)}{w(s)}\geqq 1.$$

These statements remain true for a general set S if lim sup *and* lim inf *are replaced by* sup *and* inf *respectively throughout.*

If a T obeying the conditions exists, then for any f and $\varepsilon>0$ there is an element $s(f,\varepsilon)$ after which

$$\frac{\|TV(s)f\|_B}{w(s)\|T_1\|_B}\geqq 1-\varepsilon,\qquad \frac{\|V(s)\|_A}{v(s)\|f\|_A}\geqq 1+\varepsilon,$$

so that

$$\frac{\|Tf\|_B}{\|f\|_A}\leqq\frac{\|TV(s)f\|_B}{V(s)\|f\|_V}\,\frac{v(s)}{w(s)}\,\frac{1+\varepsilon}{1-\varepsilon}\leqq T\,\frac{1+\varepsilon}{1-\varepsilon}\,\frac{v(s)}{w(s)}.$$

We can choose f so that the lefthand side is greater than $1-\varepsilon\|T\|$ and this leads to (3. 3).

With Theorem 5, this leads to the following criteria.

Theorem 7. *Let $E(\lambda, h)$, $E(\mu, h)$ be, respectively, upper and lower E functions for λ and μ. In order that a nonzero continuous solution of* (3. 1) *exist it is necessary that*

$$\text{(3. 4)} \qquad \liminf \frac{E(\lambda, h)^{1/p}}{E(\mu, h)^{1/q}} \geqq 1, \quad \liminf \frac{(1+E(\lambda, h))^{1/p}}{(1+E(\mu, h))^{1/q}} \geqq 1,$$

$$\text{(3. 5)} \qquad \liminf \frac{(E(\lambda, h)+E(\mu, -h))^{1/p}}{(E(\mu, h)+E(\mu, -h))^{1/q}} \geqq 1$$

as h tends to infinity in any direction.

These result follow on taking $V(h)$ and $W(h)$ to be $\tau(h)$, $1+\tau(h)$, $\tau(h)+\tau(-h)$ respectively, and then applying Theorems 5 and 6.

Important cases are those with $\lambda(x) = e^{a\|x\|}$ (or $(1+\|x\|)^a$) and $\lambda(x) = e^{b\|x\|}$(or $(1+\|x\|)^b$). $E(\lambda, h)$ and $E(\mu, h)$ can then be taken to be $\lambda(h)$ and $\mu(h)$ respectively; and we find that for a solution it is necessary that $qa-pb \geqq 0$, using the first inequality in (3. 4). If $a=b=0$ the first inequality gives no result: the second then shows that we must have $p \leqq q$, a result due to Hörmander ([5], Theorem 1. 1). If $qa-pb = 0$, (3. 5) shows that we must have $p \leqq q$.

Sufficiency of the conditions. The conditions given in Theorem 7 are not usually sufficient for the existence of solutions. Somewhat stronger conditions are sufficient if $p \geqq q$, as the following theorem shows.

Theorem 8. *In order that the identity be a continuous imbedding of $L(p, \lambda)$ into $L(q, \mu)$ it is necessary and sufficient that $p \geqq q$ and that $\lambda^{-q}\mu^p \in L^{1/(p-q)}$.*

For the imbedding to be continuous we must have

$$\text{(3. 6)} \qquad \left(\int f^q \mu d\chi\right)^{1/q} \leqq K\left(\int f^p \lambda d\chi\right)^{1/p}$$

for all f and some fixed constant K.

Take $f(x)=\|x\|^{-\alpha}\chi_R(x)$, where χ_R is the characteristic function of the ball $B(0, R)$. Then if $p<k/\alpha<q$ the lefthand side of (3. 6) is infinite, and the righthand side finite; hence for (3. 6) to hold it is necessary that $p \geqq q$. Now take $f=(\mu/\lambda)^{1/(p-q)}\chi_R$. Substituting in (3. 6) gives

$$\int\limits_{\|x\|<R} (\lambda^{-q}\mu^p)^{1/(q-p)}\, dx \leqq K^{\frac{pq}{p-q}},$$

driving the necessity of the second condition.

On the other hand, if the conditions hold, then by Hölder's inequality, for any $f\in L(p, \lambda)$

$$\left(\int f^q \mu dx\right)^{1/q} \leqq \left(\int f^p \lambda dx\right)^{1/p} \left(\int \mu^{p/(p-q)} \lambda^{-q/(p-q)} dx\right)^{\frac{p-q}{p}},$$

so that the conditions are sufficient.

If $\lambda(x)=e^{a\|x\|}$, $\mu(x)=e^{b\|x\|}$, the condition becomes $qa-pb > 0$. If $\lambda(x) = (1+\|x\|)^a$, $\mu(x) = (1+\|x\|)^b$, they become $qa-pb > k(p-q)$.

The study of the sufficiency of the conditions in other cases depends on examining more complicated transforms such as the Hilbert transforms or Riesz potentials.

If T is a bounded translation invariant operator from $L(p, \lambda)$ to $L(q, \mu)$ then there is a unique distribution $k\in D'$ such that

$$Tu = k*u, \quad u\in D.$$

If $\lambda(x)=O\|x\|^m$ as $\|x\| \to \infty$, for some m, then k is in S'.

Here D is Schwartz' space of infinitely differentiable functions with compact support, S his space of functions with the seminorms $\sup \{\|x\|^r |D^j f(x)|\}$, and D', S' the corresponding duals. The argument is close to that of Hörmander, Theorem 1. 2; modified to allow for the fact that S need not be in $L(p, \lambda)$.

For any $u\in D$ and any differential operator D^j, $TD^j u=D^j Tu$. Since $D^j u\in L(p, \lambda)$, $TD^j u\in L(q, \lambda)$ and so by the Sobolev imbedding theorem (cf. [5] Lemma 1. 1) $Tu(x)$ is continuous after correction on a set of measure zero and (after correction)

$$Tu(0) \leqq c \sum_{|j|\leqq k} \Big(\int_{\|x\|<1} |D^j u|^p dx\Big)^{1/p} \leqq c' \sum_{j\leqq k} \|D^j u\|_{p, \lambda}$$

for some constants C, C'. If u tends to 0 in the distribution sense, so does $\|D^j u\|_{p,\lambda}$ and hence $u \to Tu(0)\in D'$; it follows, that for some distribution k, $Tu(0)=(k * u)(0)$ and hence, by translation invariance, that $Tu=k * u$.

If, for some m, $\lambda(x)=O(\|x\|^m)$ then if $u\in S$, $D^j u\in L(p, \lambda)$ for any j, and the argument above goes through with S and S' replacing D and D'.

4. Scope of a transformation

The map $u_g: f\to \langle f, g\rangle=\int fg\,dx$ is an element of the dual of $L(p, \lambda)$ if $g\lambda^{-1}$ is in $L(p', \lambda)$ and its norm is the norm of $g\lambda^{-1}$ in that space, that is $\|u(g)\| = \left(\int |g|^{p'} \lambda^{-p'/p} dx\right)^{1/p'}$. Writing $p'' = 1/(p-1) = p'/p$, we see that g is an element of $L(p', \lambda^{-p''})$ and that $\|u(g)\|$ is the norm of g as an element of that space.

Now let T map $L(p, \lambda)$ to $L(q, \mu)$. The dual of the latter space is represented by elements $h \in L(q', \mu^{-q'})$, and for such an h $\langle Tf, h\rangle$ is continuous in f, so that there is a $T'h$ in $L(p', \lambda^{-p'})$ such that $\langle Tf, h\rangle = \langle f, T'h\rangle$.

T is represented by a distribution in D': $Tf = kf$ if $f \in D$. We have already pointed out that T is in S' if S is contained in $L(p, \lambda)$, that is to say, if $\lambda(x)$ is of not more than polynomial growth as $\|x\| \to \infty$. The result we have just proved enables us to show that this also holds if $\mu(x)^{-1}$ is of not more than polynomial growth.

If f and h are in D then

$$\langle Tf, h\rangle = \langle k * f, h\rangle = \langle f, \check{k} * h\rangle$$

where $\check{k}(x) = k(-x)$. It follows that $h \to \check{k} * h$ maps $L(q', \mu^{-q'})$ to $L(p', \lambda^{-p'})$ and if $\mu^{-q'}$ is of not more than polynomial growth $\check{k}$ and so k is in S'.

We sum up and extend these results in the following theorem.

Theorem 9. *Let T be a continuous transformation linear from $L(p_0, \lambda_0)$ to $L(q_0, \mu_0)$ with norm M_0 and from $L(p_1, \lambda_1)$ to $L(q, \mu_1)$ with norm M_1. For $0 \leqq t \leqq 1$ let*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}, \quad \lambda_t^{1/p_t} = \lambda_0^{(1-t)/p_0} \lambda_1^{t/p_1}$$

$$\mu_t^{1/p_t} = \mu_0^{(1-t)/p_0} \mu_t^{1/p_1}.$$

Then T is a continuous translation invariant map from $L(p_t, \lambda_t)$ to $L(q_t, \mu_t)$ and from $L(q_t', \mu^{-q_t''})$ to $L(p_t', \lambda^{-p_t''})$ with norm not greater than $M_0^{1-t} M_1^t$.

If, for any t, λ_t and μ_t or their reciprocals grow at infinity not faster than a polynomial, then T is represented by a convolution with a distribution k in S'.

This theorem follows from the previous arguments together with the theorem of Stein and Weiss on interpolation of operators with change of measure [9]. The importance of the last result is that it extends the range of transformations to which the Fourier transform methods of the next section apply.

5. Fourier transforms of solutions

Explicit characterizations of transforms can be obtained by Fourier transform methods for transforms acting between spaces on both of which $\|\tau(h)\|$ is a continuous function of h and τ is either compactly approximated or inversely compactly approximated on both. Theorem 4 shows that in that case $\lambda(x)$ and $\mu(x)$ are approximated by functions of the form $\exp \pm F(x)$ up to multiples of order $\exp \varepsilon \|x\|$. We shall suppose in what follows that λ and μ are exactly of these forms. The results that follow apply without change if λ and μ are of this form up to multiplication by functions that are bounded above and below by positive constants. In the general case the arguments need some modifications, but will go through if the hypotheses in Theorem 10 that $C(\mu)/q$ is contained in $C(\lambda)/p$ is replaced by the hypothesis that $C(\mu)/q$ is in the interior of $C(\lambda)/p$.

Let us suppose that $\lambda(x)=\exp F(\lambda, x)$, $\mu(x)=\exp F(\lambda, x)$, where $F(\lambda, x)$ and $F(\mu, x)$ are as in section 2. These functions are positive homogeneous and convex, and hence are the support functions of closed convex sets $C(\lambda)$, $C(\mu)$ respectively: $F(\lambda, x)=\sup\{v\cdot x\colon v\in C(\lambda)\}$.

If $f\in L(p,\lambda)$ then its Fourier transform for $w=u+iv$ is given by

$$(6.1)\qquad (2\pi)^{\frac{1}{2}k}\hat{f}(w)=\int f(x)\,e^{-iw\cdot x}\,dx=\int f(x)\,e^{v\cdot x-iu\cdot x}\,dx$$

Now

$$\int |f(x)|^p\,e^{pv\cdot x}\,dx=\int |f|^{pev\cdot x-F(\lambda,x)}\,\lambda(x)\,dx$$

so that $f(x)e^{v\cdot x}$ is in L_p if $pv.x\leqq F(\lambda, x)$ for all x, that is, if v is in $C(\lambda)/p$. If v is an interior point of $C(\lambda)/p$ let the distance of v from the exterior $C(\lambda)/p$ be $a>0$, so that $pv\cdot x-F(\lambda,x)\leqq a$ for all x; then $f(x)e^{v\cdot x}\in L$, for

$$\int |f(x)|\,e^{v\cdot x}\,dx=\int |f(x)|\,e^{v\cdot x-F(\lambda,x)/p}\,\lambda(x)^{1/p}\,dx\leqq$$

$$\leqq \|f\|_{p,\lambda}\Big[\int e^{-p'a\|x\|}\,dx\Big]^{1/p'}=C\,a^{-k/p'},$$

by Hölder's inequality, with C a constant depending only on k and p.

$\hat{f}(w)$ therefore exists if $v\in C(\lambda)/p$ whenever $p\leqq 2$ and, as a function of u, $\hat{f}(u+iv)$ is the Fourier transform of a function in L_p, that is, is in $\hat{L}_p$. In general $\hat{f}(w)$ is analytic for v in the interior of $C(\lambda)/p$ and is the Fourier transform of a function in L when considered as a function of u, and has a supremum for fixed v of the order of $a^{-k/p'}$ with a the distance of v from the boundary of $C(\lambda)/p$.

Now let $T\colon f\to k*f$ be a map from $L(p,\lambda)$ to $L(q,\mu)$; then the adjoint T' can be represented as a map $f\to k*f$ from $L(q',\mu^{-q''})$ to $L(p',\lambda^{-p''})$ and k is a distribution in S', e_w, where $e_w(x)=e^{-iw\cdot x}$, is in $L(q',\mu^{-q''})$ if $q'v.x-q''F(\mu,x)\leqq 0$ for all x, that is provided that $qv\in C(\mu)$, and then $(k*e_w)(x)=e^{-iw\cdot x}(2\pi)^{\frac{1}{2}k}\hat{k}(w)$ is in $L(p',\lambda^{-p''})$. This implies that $\hat{k}(w)$ exists and that $v\in C(\lambda)/p$. Thus for a nonzero map T to exist we must have that $C(\lambda)/q\subset C(\mu)/p$ and $\hat{k}(w)$ must be analytic for v in the interior of $C(\mu)/w$.

The problem of maps from spaces with measures of form $\exp(-F(\lambda,x))$ to those with measures of similar form reduces to the one just discussed: this is, in the notation above, the question of a map from $L(p,\lambda^{-1})$ to $L(q,\mu^{-1})$ and this is equivalent to a map from $L(q',\mu^{q''})$ to $L(p',\lambda^{p''})$; the condition for this is that $p''C(\lambda)/p'\subset$ $\subset q''C(\mu)/q'$, that is, that $C(\lambda)/p\subset C(\mu)/q$.

We now investigate conditions for a multiplier function to give such a map.

Definition. For $a \geqq 0$ let $M^q_{p,a}$ be the set of all $m(u)$ such that $m(u)\hat{f}(u)$ is in L_q whenever f is in $L(p, e^{a\|x\|})$. If $m(u)\hat{f}(u)=\hat{g}(u)$ then we write $M^q_{p,a}(m)$ (or $M^q_{p,a}(m(u))$ when it is necessary to specify the variable in m) for the norm of the map $f \to g$ on $L(p, e^{a\|x\|})$, to L_q.

Since $L(p, e^{a\|x\|})$ decreases as a increases, and since the norm of a fixed element increases with a, $M^q_{p,a}$ increases with a, and the norm of a fixed m decreases with a. These monotonicities are strict. Thus in particular if $p>q$, $M^q_{p,a}$ is empty if $a=0$, but contains the unit function if $a>0$.

Theorem 10. *Let $\lambda(x)=\exp F(\lambda, x)$, $\mu(x)=\exp F(\mu, x)$, where $F(\lambda, x)$, $F(\mu, x)$ are support functions of closed convex sets $C(\lambda)$, $C(\mu)$. Let $pF(\lambda, x) \geqq qF(\mu, x)$ for all x. Let $m(w)$ be analytic if $v \in C(\mu)/q$ and for each v in $C(\mu)/q$ let $m(u+iv)$ be in $M^q_{p,\delta}$ with $M^q_{p,\delta}(m(u+iv)) \leqq K\delta^{-\gamma}$ where K, γ are constants independent of v, $\gamma<k$, and δ is the distance of v from the boundary of $C(\lambda)/p$. Let $C(\mu)/q$ be not entirely contained in the boundary of $C(\lambda)/p$. Then if $q \leqq 2$ the map $f \to g$, where $\hat{g}(w)=m(w)\hat{f}(w)$ for $v \in C(\mu)/q$ is a bounded translation invariant map from $L(p, \lambda)$ to $L(q, \mu)$.*

If $2<q<\infty$ the same holds provided that $m(w)$ is uniformly bounded for v in $C(\mu)$.

According to the hypotheses there is for each v in $C(\mu)/q$ an element $g(v, x) \in L_q$ such that

$$(6.3) \qquad (2\pi)^{\frac{1}{2}k}\hat{g}(u+iv) = \int g(v, x)\, e^{-iu \cdot x}\, dx,$$

and

$$(6.4) \qquad \|g(v, \cdot)\|_q \leqq \delta^{-\gamma} K \left(\int |f(x)|^p e^{\delta\|x\|}\, dx\right)^{1/p}.$$

Our first aim is to prove that this g is essentially independent of v, that is, $g(v, x)= =g(x)e^{v \cdot x}$ for some g in $L(q, \mu)$.

Choose a, b, v in $C(\mu)/q$ so that $a_r \leqq v_r \leqq b_r$ for all r; we may as well suppose the inequalities strict, since there is nothing to prove unless some are strict and we can ignore the coordinates for which a_r, b_r, v_r are equal. Letting $w_r = u_r + iv_r$, for large enough T Cauchy's theorem gives

$$(6.5) \qquad \hat{g}(w) = \int_{C(T)} \frac{\hat{g}(z)\, dz}{P(z, w)},$$

where $P(z, w) = (2\pi i)^{-k} \Pi (z_r - w_r)$, and where $C(T) = \Pi C_r(T)$ with $C_r(T)$ the rectangle with vertices at $ia_r \pm T$, $ib_r \pm T$ described positively. Calling the integral on the right $I(T)$ it follows that

$$\hat{g}(w) = \int_T^{T+1} I(t)\, dt = J(T) + R(T),$$

where $J(T)$ is the part of the integral in (6. 4) over the product of the horizontal sides of the rectangles, and $R(T)$ is a sum of terms of the forms

$$\int_T^{T+1} dt \int_{ia+T}^{ib+T} \frac{\hat{g}(z)}{P(z, w)} dz \quad \text{and} \quad \int_T^{T+1} dt \int_T^{t} \frac{\hat{g}(ia+x)}{P(ia+x, w)} dx$$

together with others obtained by replacing T and $T+1$ by $-T$, $-T-1$ and a by b. The terms of the first type are not greater in modulus than

(6. 6)

$$\int_a^b dy \int_T^{T+1} \frac{|g(t+iy)|}{|P(t+iy, w)|} dt \leqq \int_a^b dy \left(\int_T^{T+1} \hat{g}(t+iy)|^{q'} dt \right)^{1/q'} \left(\int_T^{T+1} \frac{dt}{|P(t+iy, w)|^q} \right)^{1/q} \leqq$$

$$\leqq AT^{-k} \int_a^b dy \left(\int_T^{T+1} |\hat{g}(t+iy)|^{q'} dt \right)^{1/q'},$$

for some constant A. If $q \leqq 2$ the inner integral is not greater than $C_q \| g(y, \cdot) \|_q$ where C_q is the norm of the Fourier transform as a map from L_q to $L_{q'}$. Moreover, $\hat{g}(u+iv) = m(u+iv)\hat{h}(u)$ where $k(x) = f(x)e^{v \cdot x}$ and then

$$\int |h(x)|^p e^{\delta \|x\|} dx = \int |f(x)|^p e^{pv \cdot x + \delta x} dx \leqq \int |f(x)|^p e^{F(\lambda, x)} dx,$$

because $F(\lambda, x) = \sup \{v.x : v \in C(\lambda)\} \geqq pv.x + \delta \|x\|$ since all points within δ of v are in $C(\lambda)$. Then by (6. 4) $\|g(v, \cdot)\|_q \leqq K\delta^{-\gamma} \|f\|_p$.

We can suppose without loss of generality that at most one point of $[a, b]$, say a, lies in the boundary of $C(\lambda)/p$. The integral in (6. 6) is of the order of $\int_a^b \delta^{-\gamma} dy$ where δ is the distance of y from the boundary of $C(\lambda)/p$. If a is not in the boundary, this is bounded; if a is in the boundary, it is of the order of the integral $\int_0 r^{k-1-\gamma} dr$ with $r = \|y-a\|$, and is again bounded if $\gamma < k$ as required by hypothesis. Hence the term (6. 6) is $O(T^{-k})$ as $T \to \infty$.

This argument fails if $q < 2$; however, in that case under the strengthened hypothesis that $m(u+iv)$ is bounded we can consider the side terms in the integral (6. 5) directly. According to (6. 2), assuming again that a is in the boundary of $C(\lambda)/p$, $|f(u+iv)|$ is bounded by a constant multiple of $\delta^{-k/p'}$ and, arguing much as above, the integral is of the order of T^{-k} multiplied by the integral of $\int_0 r^{k(1-1/p')-1} dr$ and since $p' > 1$ this is bounded. Once again the side terms tend to 0 and we can write $\hat{g}(w) = I_a - I_b$ where

$$(2\pi)^{\frac{1}{2}k} I_a = \int\limits_{y=a} \frac{dz}{P(z, w)} \int\limits_{-\infty}^{\infty} g(a, \xi) e^{-ix\cdot\xi} d\xi =$$

$$= \int\limits_{-\infty}^{\infty} g(a, \xi) d\xi \int\limits_{y=a} \frac{e^{-ix\cdot\xi}}{P(z, w)} dz = \int\limits_{0}^{\infty} g(a, \xi) e^{-a\cdot\xi - i\xi\cdot w} d\xi,$$

and similarly

$$(2\pi)^{\frac{1}{2}k} I_b = -\int\limits_{-\infty}^{0} g(b, \xi) e^{-b\cdot\xi - i\xi\cdot w} d\xi.$$

On comparison with (6. 3), it follows that $g(x) = g(v, x)e^{-v\cdot x}$ is independent of v and then

$$\left(\int |g(x)|^q \mu(dx)\right)^{1/q} = \left(\int |g(v, x)|^q e^{-qv\cdot x + F(\mu, x)} dx\right)^{1/q} \leqq$$

$$\leqq \left(\int |g(v, x)|^q dx\right)^{1/q} \leqq K\delta^{-\gamma} f_p,$$

when the point v is chosen arbitrarily in the interior of $C(\lambda)/p$. Thus the map $f \to g$ is continuous on $L(p, \lambda)$ to $L(q, \mu)$.

Note that the conclusions of the theorem are valid if $q \leqq 2$ even if $C(\mu)/q$ is entirely contained in the boundary of $C(\lambda)/p$ provided that γ can be taken to be zero.

Sufficient conditions for a map generated by m to be continuous $L(p, \lambda^{-1}) \to L(q, \mu^{-1})$ follow from this theorem, by using the duality arguments. These are that $C(\lambda)/p \subset C(\mu)/q$, that $C(\lambda)/p$ is not completely contained in the boundary of $C(\mu)/q$, that if $p \geqq 2m(u+iv)$ is in $M^{p'}_{q',\delta}$ and that $M^{p'}_{q',\delta}(m(u+iv)) < K\delta^{-\gamma}$ where δ is the distance from v to the boundary of $C(\mu)/q$ and K, γ are as before. If $p < 2$ we need in addition that $m(u+iv)$ is uniformly bounded for $v \in C(\mu)/q$.

These conditions are unaltered by translations of $C(\lambda)/p$, $C(\mu)/q$ through the same displacement. This is a particular case of the following observation:

If the kernel k generates a continuous transformation from $L(p, \lambda)$ to $L(q, \mu)$ then the kernel $k(x)e^{a\cdot x}$ generates a continuous transformation from $L(p, \lambda e^{-pa\cdot x})$ to $L(q, \mu e^{-qa\cdot x})$.

The effect of the change in measures involved in this statement is to alter $F(\lambda, x)$ to $F(\lambda, x) - pa\cdot x$ and $F(\mu, x)$ to $F(\mu, x) - qa\cdot x$ and so to displace both $C(\lambda)/p$ and $C(\mu)/q$ by $-a$.

We give some applications of these arguments to particular kernels.

a) The one-dimensional Hilbert transform has $k(x) = 1/\pi x$, $\hat{k}(u) = \operatorname{sgn} u$. This has no analytic extension, so that H cannot map any $L(p, \lambda)$ continuously to $L(q, \mu)$ if $C(\mu)$ contains any nonzero v: that is to say, if μ is in the class we are considering in this section, it must be constant.

b) The Riesz potentials are the transforms R_α with kernels $\|x\|^{\alpha-k}$, $0<\alpha<k$, apart from a constant multiple; the Fourier transform $m(u)$ is a constant multiple of $\|u\|^{-\alpha}$. This has no analytic extension and the same conclusions apply as for the Hilbert transform.

c) Let $e(x)=e^{-\|x\|}$. Then $\hat{e}(w) = (1+\sum w_r^2)^{-k}$ is analytic for $\|v\|<1$, and uniformly bounded on any region $\|v\| < 1-\delta$, if $\delta>0$. Conditions sufficient in order that the map T,

$$Tf(x) = \int e^{-|x-y|} f(y)\,dy,$$

act continuously $L(p, \lambda)\to L(q, \mu)$ are:

A. $F(\mu, x)<q\|x\|$ for all x and B: if $p\leqq q\leqq 2$, $pF(\mu, x) \leqq (qF\lambda, x)$ and if $p>q$, $pF(\mu, x)<qF(\lambda, x)$ for all x.

Dually, the conditions that T act continuously from $L(p, \mu^{-1})$ to $L(q, \lambda^{-1})$ are

A'. $F(\lambda, x)<p\|x\|$; B' if $q\geqq p\geqq 2$, $qF(\lambda, x)\leqq pF(\mu, x)$ for all x, if $p>q$ $qF(\lambda, x)<pF(\mu, x)$ for all x.

d) For the kernel $k(x)=e^{-\|x\|^2}$ the conditions are the conditions B and B' above.

e) If $q<p$ and $C(\mu)/q$ is contained in the interior of $C(\lambda)/p$ then the identity is a continuous map $L(p, \lambda)\to L(q, \mu)$.

This follows easily enough from the theorem, or directly from Theorem 8, for if the conditions hold then there is an ε neighbourhood of $C(\mu)/q$ in $C(\lambda)/p$ and so $F(\lambda, x)/p-F(\mu, x)/q > \varepsilon\|x\|$ for all x, so that the conditions of Theorem 8 hold.

The theorem also allows us to state some cases in which the class of translation invariant maps from $L(p, \lambda)$ to $L(q, \mu)$ is vacuous, that is, consists only of the zero map. Among these are the following:

a) $C(\mu)/q$ is not in $C(\lambda)/p$, that is, $pF(\mu, x)>qF(\lambda, x)$ for some x.

b) $p>q$ and $C(\mu)/q$ contains a boundary point of $C(\lambda)/p$, that is $pF(\mu, x)= =qF(\lambda, x)$ for some x.

For suppose that there is a v common to the boundaries; for this v, $\delta=0$, and if $m(w)$ generates a map $L(p, \lambda)\to L(q, \mu)$ then $m(u+iv)\in M_p^q$ and this, as we have seen, is vacuous if $p>q$.

On the other hand, the class of maps is never vacuous if $C(\mu)/q$ is in the interior of $C(\lambda)/p$; for then if $p>q$ the identity is a continuous map $L(p, \lambda)\to L(q, \mu)$ and if $p\leqq q$ the function $m(w)=e^{-w^2}$ is a multiplier generating a nontrivial continuous map.

Lastly, with λ and μ of the same forms, there remains the question of the existence of map $L(p, \lambda)\to L(q,\mu^{-1})$ and $L(p, \lambda^{-1})\to L(q, \mu)$. Here the functions in one space or the other are very large at infinity, and do not have Fourier transforms: nor do the functions involved in the dual problem in the other space. The problem can be considered by other methods, and some results may be obtained by using generalized Fourier transforms. However, the following result is obvious, as consequences of previous theorems.

For the map $L(p, \lambda^{-1}) \to L(q, \mu)$ the comparison functions of Theorem 7 are $E(\lambda^{-1}, h) = -\exp(-F(\lambda, h))$, $E(\mu, h) = \exp F(\mu, h)$ so that $L(p, \lambda^{-1}) \to L(q, \mu)$ is vacuous unless λ and μ are 1.

The question of maps from $L(p, \lambda)$ to $L(q, \mu^{-1})$ reduces to that of maps from $L(p, \lambda^{e^{pa \cdot x}})$ to $L(q, \mu^{-1} e^{qa \cdot x})$ according to the argument above, where a is any point. If a is common to $C(\lambda)/p$ and $-C(\mu)/q$, 0 is a common point of the corresponding regions after displacement: for the equivalent F's $F(\lambda, x) \geqq 0$, $F(\mu, x) \geqq 0$ for all x. Assuming this to be the case, we have the continuous imbeddings $L(p, \lambda) \subset L_p$, $L_q \subset L(q, \mu^{-1})$, and so any continuous map $L_p \to L_q$ induces one from $L(p, \lambda)$ to $L(q, \mu^{-1})$. If $p \leqq q$ there are always such maps that are not zero. If $p > q$, the identity is a map from $L(p, \lambda)$ to $L(q, \mu^{-1})$ if a is interior to $C(\lambda)/p$.

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