A radical class which is fully determined by a lattice isomorphism

By B. DE LA ROSA in Bloemfontein (South Africa)*)

In this note we introduce the concept of a quasi-semi-prime ideal in an associative ring as a generalization of the notion of a semi-prime ideal. We consider the class of rings in which all ideals have this property. This class is shown to be a non-hereditary radical class in the sense of Kurosh. As an application we show that the existence of a lattice isomorphism between the lattice of (two-sided) ideals in a ring R and the lattice of ideals in the ring R_n of $n \times n$ matrices over R, is equivalent to the fact that R belongs to the mentioned radical class.

1. The λ-radical of a ring

Definition 1. An ideal Q in a ring R may be called a *quasi-semi-prime ideal* if from $RAR \subseteqq Q$, where A is an ideal in R, it follows that $A \subseteqq Q$.

The following two theorems are easy consequences of this definition.

Theorem 2. *For an ideal Q of a ring R the following statements are equivalent:*

(i) *Q is a quasi-semi-prime ideal in R.*

(ii) *If a is an element of R such that $RaR \subseteqq Q$, then $a \in Q$.*

Theorem 3. *For a ring R the following statements are equivalent.*

(i) *All ideals in R are quasi-semi-prime.*

(ii) *Each element a of R belongs to the corresponding ideal RaR.*

(iii) *Each ideal A of R satisfies the relation $RAR = A$.*

We shall make particular use of the second condition of the latter theorem. For convenience we introduce the following terminology: An element a of a ring R is called a λ-element if $a \in RaR$. A ring R is called a λ-ring if every element of R is a λ-element. An ideal A of a ring is called a λ-ideal if A is a λ-ring. A ring is said to be λ-semi-simple if it contains no non-zero λ-ideals. We show that the class of

*) The author gratefully acknowledges financial assistance received from the G.S.I.R., South Africa, during his stay in Delft.

λ-rings is a radical class according to the definition by KUROSH as adapted by DIVINSKY (cf. [2]).

Firstly, we note that the homomorphic closure of this class of rings follows directly by applying the operation preserving properties of ring homomorphisms.

Secondly, it is obvious that the zero ideal of an arbitrary ring is a λ-ideal. To show that every ring contains a unique maximal λ-ideal, we verify the following

Lemma 4. *The union $\lambda(R)$ of all the λ-ideals of a ring R is a λ-ideal in R.*

Proof. Let A and B be λ-ideals in R and let s be an arbitrary element of $A+B$. Then $s=a+b$, where $a\in A$ and $b\in B$. Since A is a λ-ideal in R, there exist elements x_i, y_i in A such that $a=\Sigma x_i a y_i$. Denoting the element $\Sigma x_i(a+b)y_i$ by c, we can write:

$$a+b-c=a+b-\Sigma x_i(a+b)y_i=b-\Sigma x_i b y_i.$$

Hence it follows that $a+b-c\in B$. Since B is a λ-ideal in R, there exist elements u_j, v_j in B such that

$$a+b-c=\Sigma u_j(a+b-c)v_j.$$

It follows that

$$a+b=\Sigma x_i(a+b)y_i+\Sigma u_j(a+b)v_j-\Sigma u_j[\Sigma x_i(a+b)y_i]v_j.$$

Since clearly $x_i, y_i, u_j, v_j, u_j x_i, y_i v_j\in A+B$, we have that s is a λ-element in $A+B$. Therefore $A+B$ is a λ-ideal in R.

Finally, since each element of the union of all λ-ideals of R belongs to the sum of a finite number of these ideals, it is clear that every such element is a λ-element in $\lambda(R)$. Therefore $\lambda(R)$ is a λ-ideal in R. This completes the proof of the lemma.

There remains to show that the factor ring $R/\lambda(R)$ is λ-semi-simple.

Lemma 5. *The factor ring $R/\lambda(R)$ contains no non-zero λ-ideals.*

Proof. Let $H/\lambda(R)$ be a λ-ideal in $R/\lambda(R)$ and let $h+\lambda(R)$ be an arbitrary element of $H/\lambda(R)$. Then there are elements x_i, y_i in H such that

$$h+\lambda(R)=\Sigma\big(x_i+\lambda(R)\big)\big(h+\lambda(R)\big)\big(y_i+\lambda(R)\big)=\Sigma x_i h y_i+\lambda(R).$$

This implies that $h-\Sigma x_i h y_i\in\lambda(R)$, and since $\lambda(R)$ is a λ-ideal in R, it follows that $h-\Sigma x_i h y_i=\Sigma u_j(h-\Sigma x_i h y_i)v_j$ for some $u_j, v_j\in\lambda(R)$. Therefore

$$h=\Sigma x_i h y_i+\Sigma u_j h v_j-\Sigma u_j[\Sigma x_i h y_i]v_j.$$

Since $u_j, v_j\in\lambda(R)\subseteqq H$, it follows that $x_i, y_i, u_j, v_j, u_j x_i, y_i v_j\in H$. The last equality therefore shows that H is a λ-ideal in R, and accordingly it must be contained in $\lambda(R)$. Therefore $H=\lambda(R)$, and $H/\lambda(R)$ is the zero ideal in $R/\lambda(R)$. This completes the proof of the lemma.

Theorem 6. *The property λ is a radical property.*

Being a radical property, λ satisfies the relation $\lambda(I) \subseteq I \cap \lambda(R)$ for an arbitrary ideal I in any ring R. The reverse inclusion, however, does not hold in general; for instance, $E \cap \lambda(Z) = E \subsetneqq \lambda(E) = 0$, where E denotes the ring of even integers and Z the ring of all integers. Thus it follows that λ *is not hereditary.*

We conclude this section by comparing the λ-radical property with those between the Baer—McCoy radical property β and the upper radical property φ determined by the class of all fields.

Theorem 7. *The λ-radical property is independent of all radical properties χ such that $\beta \leqq \chi \leqq \varphi$.*

Proof. Every ideal of a λ-radical ring R is quasi-semi-prime. Therefore $R^3 \subseteq R^2$ implies that $R \subseteq R^2$, so that $R^2 = R$. Thus it follows that a nilpotent ring is not λ-radical, and consequently $\beta \nleqq \lambda$. On the other hand all fields are χ-semi-simple and at the same time λ-radical, so that $\lambda \nleqq \chi$. The proof is completed.

This independence was to be expected since the Baer—McCoy radical, for instance, is a measure for the presence of nilpotent ideals in a ring, while the λ-radical measures the presence of "well-behaved" ideals such as regular ideals and simple non-trivial ones. Where semi-simplicity with respect to χ is of special interest from a structural point of view, the emphasis must therefore be placed on radicality with respect to λ. The following section deals with an application in this respect.

2. Rings of $n \times n$ matrices over a ring

Although the ring R under consideration needs not possess a unit element, we still use the matrix units E_{ij} in a formal way: If $x \in R$, then xE_{ij} is to be interpreted as the matrix in R_n with the element x at the intersection of the i^{th} row and j^{th} column and the zero element of R elsewhere.

Theorem 8. *An ideal Q in a ring R is quasi-semi-prime if and only if Q_n is a quasi-semi-prime ideal in R_n.*

Proof. Suppose that Q is a quasi-semi-prime ideal in R and let $\alpha = \Sigma a_{ij} E_{ij}$ be any element of R_n such that $R_n \alpha R_n \subseteq Q_n$. If $\alpha \notin Q_n$, then $a_{km} \notin Q$ for some $k, m \in \in \{1, 2, \ldots, n\}$. Since Q is a quasi-semi-prime ideal in R, we have that $Ra_{km} R \subsetneqq Q$, that is, there exist elements x and y in R such that $xa_{km} y \notin Q$. But if this was the case it would follow that $(xE_{kk})\alpha(yE_{mm}) = xa_{km} yE_{km} \notin Q_n$. However, this is impossible, since $R_n \alpha R_n \subseteq Q_n$. Therefore $\alpha \in Q_n$, and we have that Q_n is quasi-semi-prime in R_n.

Conversely, suppose that Q_n is a quasi-semi-prime ideal in R_n, and let $a \in R$ such that $RaR \subseteq Q$. We shall show that $R_n \gamma R_n \subseteq Q_n$, where $\gamma = \Sigma aE_{ij}$. An arbitrary element of $R_n\gamma R_n$ is a finite sum of elements of the form

$$\gamma' = (\Sigma x_{ij}E_{ij})(\Sigma aE_{ij})(\Sigma y_{ij}E_{ij}),$$

which is the sum of $n\times n$ matrices of the form $x_{pr}ay_{sq}E_{pq}$. Since $RaR\subseteqq Q$, it follows that $x_{pr}ay_{sq}\in Q$, and hence $\gamma'\in Q_n$. Therefore $R_n\gamma R_n\subseteqq Q_n$. Since Q_n is a quasi-semi-prime ideal in R_n, it follows that $\gamma=\Sigma aE_{ij}\in Q_n$, and thus that $a\in Q$. Therefore Q is a quasi-semi-prime ideal in R. This completes the proof of the theorem.

To prove our final result, we shall need the following fact (cf. [3]).

Lemma 9. *If $\mathcal{M}$ is an ideal in the ring R_n then the set M of all elements at the intersections of the first rows and first columns of matrices in $\mathcal{M}$ is an ideal in R.*

Theorem 10. *The ideals of the ring R_n are of the form M_n, where M is an ideal in R, if and only if R is a λ-radical ring.*

Proof. Suppose that R is a λ-radical ring. Let $\mathcal{M}$ be an arbitrary ideal in R_n, and let M be the ideal in R associated with $\mathcal{M}$ as in Lemma 9. We show that $\mathcal{M}=M_n$. Let $\alpha=\Sigma a_{ij}E_{ij}\in\mathcal{M}$. Then, for arbitrary $x, y\in R$, one has

$$xa_{rs}yE_{11} = (xE_{1r})(\Sigma a_{ij}E_{ij})(yE_{s1})\in\mathcal{M}.$$

Thus, by definition of M, we have that $xa_{rs}y\in M$. Since this is true for arbitrary $x, y\in R$, it follows that $Ra_{rs}R\subseteqq M$, and the fact that M is a quasi-semi-prime ideal in R ensures that $a_{rs}\in M$. The latter relationship, being true for all $r, s\in\{1, 2, \dots, n\}$, yields the fact that $\alpha\in M_n$. Therefore $\mathcal{M}\subseteqq M_n$.

If, on the other hand, m is any element of M, then there exists a matrix $\Sigma m_{ij}E_{ij}$ in $\mathcal{M}$ with $m_{11}=m$ and it follows that

$$xm_{11}yE_{pq} = (xE_{p1})(\Sigma m_{ij}E_{ij})(yE_{1q})\in\mathcal{M},$$

that is, $xmyE_{pq}\in\mathcal{M}$, where x and y are arbitrary elements of R and $p, q\in\{1, 2, \dots, n\}$. Thus every finite sum of the form $\Sigma x_i my_i E_{pq}$, $(x_i, y_i\in R)$, belongs to $\mathcal{M}$. Since R is λ-radical, it follows that $mE_{pq}\in\mathcal{M}$ for every $m\in M$. Consequently $M_n\subseteqq\mathcal{M}$, and we have that $\mathcal{M}=M_n$.

Conversely, suppose that every ideal in R_n has the form M_n, where M is an ideal in R, and let A be any ideal in R. Then the sets $\mathcal{L}$ and $\mathcal{R}$ of matrices in R_n with entries running through the ideals A, RA and AR as indicated in

$$\begin{bmatrix} A & A\dots A \\ RA & RA\cdots RA \\ \dots & \dots \\ RA & RA\cdots RA \end{bmatrix} \quad\text{and}\quad \begin{bmatrix} A & AR\dots AR \\ A & AR\dots AR \\ \dots & \dots \\ A & AR\dots AR \end{bmatrix}$$

respectively, are obviously ideals in R_n. By the hypothesis on R_n it follows that

$\mathscr{L}=\mathscr{R}=A_n=(RA)_n=(AR)_n$. Hence we have $A=RA=AR$, and it follows that $RAR=A$. The required result follows from Theorem 3 (iii).

By the preceding two theorems we obtain the following

Corollary 11. *The ring R_n is λ-radical if and only if R is λ-radical.*

References

[1] B. Brown and N. H. McCoy, The maximal regular ideal of a ring, ***Proc. Amer. Math. Soc.***, **1** (1950), 165—171.

[2] N. J. Divinsky, *Rings and Radicals*, Univ. of Toronto Press (1965).

[3] N. H. McCoy, *The Theory of Rings*, Macmillan (New York, 1964).

(Received February 18, 1971)