Bi-ideals in associative rings and semigroups

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The concept of a quasi-ideal in an associative ring was introduced by OTTO STEINFELD in [18, 20]. He has developed an extensive theory concerning quasi-ideals in rings and semigroups. Bi-ideals were introduced in semigroups by GOOD and HUGHES [2], further treated by LAJOS [10, 12] and the author [6] among others. An explicit treatment has recently been given for bi-ideals in rings by LAJOS and SZÁSZ [13, 14]. We continue to develop some of the theory of bi-ideals in rings here.

In [20] STEINFELD showed that each minimal quasi-ideal of a ring R is either null or a division ring of the form eRe. We consider here bi-ideals of a ring and show that an analogous result is also true. In a regular ring the sets of bi-ideals and quasi-ideals coincide [10]. However as LUH points out [15] a ring may have these sets coincide without being regular. In general, a quasi-ideal is a bi-ideal. We will investigate minimal bi-ideals in arbitrary rings and determine several conditions under which such bi-ideals are quasi-ideals.

§ 1. Preliminaries, bi-ideals and regularity

We begin by recalling the following definitions for rings. For semigroups (written multiplicatively) one obtains from the following the corresponding definition of quasi-ideal, bi-ideal etc. by considering only the multiplicative requirement. In the sequel, when a definition or proposition holds for semigroups with this obvious modification we will place ($\mathcal{S}$) following the number of the statement.

We will assume that the semigroup has a zero, 0, since a zero element can always be adjoined (cf. [1] p. 4) and we will write $S=S^0$ to denote such a semigroup. When the corresponding result for semigroups is known we will cite the appropriate reference by: (1. 3) ($\mathcal{S}$-[1] p. 85 ex. 18b).

(1. 1) ($\mathcal{S}$) Definition. A subgroup $(A, +)$, of a ring, R, is a *quasi-ideal* of R if $RA \cap AR \subseteqq A$. (As usual $CD=\{\sum^{n} c_i d_i \mid c_i \in C, d_i \in D\}$ for subgroups $(C, +)$ and $(D, +)$ of a ring R.) For the semigroup S, we require $A \neq \emptyset$, the empty set. and $SA \cap AS \subseteqq A$.

(1. 2) ($\mathscr{S}$) Definition. A subring B, of a ring, R is a *bi-ideal* of R if $BRB \subseteqq B$. For the semigroup S, a non-empty subset, B, is a *bi-ideal* if $B^2 \cup BRB \subseteqq B$.

(1. 3) ($\mathscr{S}$-[1] p. 85 ex. 18b) Proposition. *Let B be a bi-ideal of a ring R. Then $J = B+BR$ and $L = B+RB$ are respectively right and left ideals of R and $JL \subseteqq B \subseteqq$ $\subseteqq J \cap L$.*

Proof. A straightforward check shows that J and L are indeed right and left ideals of R. Clearly $B \subseteqq J \cap L$. On the other hand $JL = (B+BR)(B+RB) \subseteqq B^2+$ $+BRB+BR^2B \subseteqq B$ since B is a bi-ideal and the result follows.

We have the following partial converse of (1. 3).

(1. 4) ($\mathscr{S}$-[1] p. 85 ex. 18c) Proposition. *Let J and L be respectively right and left ideals of a ring R. Then any subgroup $(B, +)$ of R such that $JL \subseteqq B \subseteqq J \cap L$ is a bi-ideal of R.*

Proof. B is already by hypothesis a subgroup of R.

Since $B^2 \subseteqq JL$ and $JL \subseteqq B$ it follows that B is a subring. Moreover $BRB \subseteqq$ $\subseteqq (J \cap L)R(J \cap L) \subseteqq JRL \subseteqq JL \subseteqq B$ so that B is indeed a bi-ideal.

Unlike the semigroup case, the additional assumption that B is a subgroup is necessary in (1. 4) as the following example shows.

(1. 5) Example. Let $R = \{\alpha | \alpha: Z \to Z$, Z the set of integers with $(n)\alpha = n(2k)$, k fixed$\}$ with the functional compositions defined in the usual fashion: $(n)[\alpha+\beta] =$ $= (n)\alpha+(n)\beta$, $(n)[\alpha \cdot \beta] = ((n)\alpha)\beta$. Let $J=L=R^2$ and $B = \{\beta \in R | \, |(n)\beta| > 4n$ for each $n \neq 0$, or $(n)\beta = 4n$, or $\beta = 0\}$. Clearly $JL \subseteqq B \subseteqq R^2 = J \cap L$ and yet with $(n)\beta = 4n$, $\beta \in B$but $-\beta \notin B$ so that B is not even a subgroup no less a bi-ideal.

In the remaining part of this section we will consider bi-ideals which are either themselves regular rings [semigroups] or which are subrings [subsemigroups] of a regular ring [semigroup].

(1. 6) ($\mathscr{S}$) Definition. An element a of a ring R is *regular* if $a \in aRa$. R is *regular* if each element in it is regular.

We now have the following proposition.

(1. 7) ($\mathscr{S}$-[11] Theorem 10) Proposition. *Let $a \in R$, a ring. Then aRa is a bi-ideal. Indeed, a is regular if and only if aRa is the smallest bi-ideal containing a.*

Proof. By [12] Theorem 8, aRa is a bi-ideal. Suppose now that a is regular. Then $a \in aRa$. Let B be a bi-ideal containing a. We have $aRa \subseteqq BRB \subseteqq B$ so that aRa is indeed the smallest bi-ideal containing a. Conversely if aRa contains a then a is regular.

(1. 8) ($\mathscr{S}$) Proposition. *Let B be a bi-ideal of a ring R. If B is itself a regular ring then any bi-ideal of B is a bi-ideal of R.*

Proof. Let A be a bi-ideal of B. Then A is also a subring of R. Since B is regular we have $A \subseteqq AB$ and $A \subseteqq BA$ so that $ARA \subseteqq (AB)R(BA) \subseteqq A(BRB)A \subseteqq ABA \subseteqq A$. Thus A is a bi-ideal of R.

The following two propositions are generalizations of [1] ex. 18d, p. 85, [6] (2. 15) and [10] Theorem 1.

(1. 9) Proposition. *Let S be a semigroup and B a bi-ideal of S. If the elements of B are regular then B is a quasi-ideal.*

Proof. If $bs = rb^* \in BS \cap SB$ then there is a $b' \in S$ such that $bb'\, b = b$. Thus $bs = bb'bs = b(b'r)b^* \in BSB \subseteqq B$. Whence $BS \cap SB \subseteqq B$ and B is a quasi-ideal.

(1. 10) Proposition. *Let R be a ring and B a bi-ideal of R. If every element of B is regular then B is a quasi-ideal.*

Proof. Let $x \in BR \cap RB$. Then $x = \sum_{i=1}^{n} b_i r_i = \sum_{j=1}^{m} s_j b_j (*)$ where $b_i, b_j \in B$, $r_i, s_j \in R$. We procede inductively: $b_1 = b_1 t_1 b_1$ for some $t_1 \in R$ so $b_1 r_1 = b_1 t_1 b_1 r_1 = -\sum_{2}^{n} b_1 t_1 b_i r_i + b_1 t_1 x = -\sum_{2}^{n} b_i' r_i + b_1'$ where $b_1' \in B$ since $x \in RB$. Substituting back in $(*)$ $b_1' + \sum_{2}^{n} b_i'' r_i = x$. Again for b_2 we have a $t_2 \in R$ with $b_2 = b_2 t_2 b_2$ so that

$b_2 r_2 = -\sum_{3}^{n} b_i'' r_i + x - b_1'$ and

$$b_2 r_2 = b_2 t_2 b_2 r_2 = -\sum_{3}^{n} b_2 t_2 b_i'' r_i + b_2 t_2 (x - b_1') = -\sum_{3}^{n} b_2 t_2 b_i'' r_i + b_2'.$$

Substituting again we have $b_1' + b_2' + \sum_{3}^{n} b_i'' r_i = x$. We continue in the above fashion to obtain $\sum_{1}^{n} b_i' = x$. Since $(B, +)$ is a group the result follows.

It is possible to combine this with (1. 8) to obtain:

(1. 11) Corollary. *Let B be a bi-ideal of a ring R. If B is itself a regular ring then any bi-ideal of B is a quasi-ideal of R as well as B. If Q is a quasi-ideal of R which is itself regular then any quasi-ideal of Q is also a quasi-ideal of R.*

§ 2. General results on minimal bi-ideals

We gather in this section several general results concerning minimal bi-ideals. We have first the following definition.

(2. 1) ($\mathscr{S}$) Definition. A non-zero quasi-ideal [bi-ideal] U of a ring R is a *minimal* quasi-ideal [bi-ideal] if there is no quasi-ideal [bi-ideal], T, with $\{0\} \subset T \subset U$.

(We use $\subset$ for proper containment.) A similar definition is given for a semigroup $S=S^0$.

(2. 2) ($\mathscr{S}$-[6] (1. 8)) Proposition. *Let B be a minimal bi-ideal of a ring R. Then B is nilpotent if and only if $B^2=\{0\}$.*

Proof. Let $n \geqq 2$. Then since the product of two bi-ideals is a bi-ideal B^{n-1} is a bi-ideal which is clearly contained in B and we have $B^{n-1}=B$ if $B^{n-1}\neq\{0\}$. Thus $B^n=B^2=\{0\}$ precisely when B is nilpotent.

(2. 3) ($\mathscr{S}$) Proposition. *Let B be a minimal bi-ideal of a ring R with $B^2\neq\{0\}$. If $b_1\ Bb_1=\{0\}$ and $b_2\ Bb_2=\{0\}$ [$b_1Rb_1=\{0\}$ and $b_2Rb_2=\{0\}$] for fixed $b_1, b_2\in B$ then $b_1\ Bb_2=\{0\}$ and $b_2\ Bb_1=\{0\}$ [$b_1Rb_2=\{0\}$ and $b_2Rb_1=\{0\}$].*

Proof. If $b_1Bb_2\neq\{0\}$ then $b_1Bb_2=B$ by the minimality of B and [12] Theorem 8. We have $B^2=(b_1Bb_2b_1)Bb_2\subseteqq(b_1Bb_1)Bb_2=\{0\}$ a contradiction. Thus $b_1Bb_2=\{0\}$ and similarly $b_2Bb_1=\{0\}$.

The proof of the alternate reading is similar. Here we would have $B^2= =(b_1Rb_2b_1)Rb_2\subseteqq(b_1Rb_1)Rb_2=\{0\}$.

We remark that the above proposition is also valid with any bi-ideal T in place of R provided either $Tb_2\subseteqq T$ or $b_1T\subseteqq T$.

(2. 4) ($\mathscr{S}$-[6] (1. 8)) Theorem. *Let B be a minimal bi-ideal of a ring R. If $B^2\neq\{0\}$ then B is a division ring and a minimal quasi-ideal. Indeed B is of the form $B=eRe=eBe$ where e is the identity of B.*

Proof. Let $C=\{b\in B\mid bB=\{0\}\}$. It follows immediately that C is a subring since B is. Moreover for $c_1, c_2\in C$, $r\in R$, $c_1rc_2\in B$ and hence $c_1rc_2\in C$ so that C is a bi-ideal of R. By the minimality of B we must have $C=\{0\}$ since $B^2\neq\{0\}$ by hypothesis. Thus for $b\in B\setminus\{0\}$, $b\ B\neq\{0\}$. Since bB is a bi-ideal ([12] Theorem 8) it follows that $bB=B$. Similarly $Bb=B$. Thus for $b\in B\setminus\{0\}$ we have $Bb=bB=B$ and it follows that B is a division ring. Clearly B is thus regular so that B is a quasi-ideal by (1. 10). Since B is minimal as a bi-ideal it is surely minimal as a quasi-ideal. It now follows immediately from [20] Theorem 3 that $B=eRe=eBe$ where e is the identity of B.

§ 3. Nilpotent minimal bi-ideals

We have seen in the last section that minimal bi-ideals which are not nilpotent are quasi-ideals and moreover division rings. We will now consider the alternative case when the bi-ideal is nilpotent (recall (2. 2)!).

(3. 1) Definition. We will call a minimal bi-ideal [quasi-ideal] B a *nilpotent minimal bi-ideal* [quasi-ideal] if B is a zero subring, i.e., $B^2=\{0\}$.

As the following example show, even in a commutative ring, a nilpotent minimal bi-ideal need not be a (minimal) quasi-ideal. Thus the sets of minimal bi-ideals and minimal quasi-ideals for a given ring need not coincide.

(3. 2) Example. Let $S=Z/(6)$ where Z is the ring of integers and set $R=R[\bar{x}]=S[x]/(x^4)$ where x is transcendental over S. Let $B=\{0, 2\bar{x}^2, 4\bar{x}^2\}$. Clearly B is a subring of R. Since $B^2=\{0\}$ and R is commutative $BRB=B^2R=\{0\}\subseteq B$ so that B is a bi-ideal of R. However $4\bar{x}^3=\bar{x}(4\bar{x}^2)=(4\bar{x}^2)\bar{x}\in BR\cap RB$ but $4\bar{x}^3\notin B$ so that B is not a quasi-ideal. It is easy to see that B is also a minimal bi-ideal.

We note that a similar statement is also true in the case of a commutative semigroup. It suffices to consider $(R, \cdot)$ above as our semigroup and $B'=\{0, 4\bar{x}^2\}$. Then $(B')^2=\{0\}$ so that $\{0\}=B'RB'\subseteq B'$ while again $4\bar{x}^3\notin B'$.

(3. 3) Theorem. *Let B be a nilpotent minimal bi-ideal of a semigroup $S=S^0$. Then the following sets of equivalent statements are mutually exclusive.*

1. *some non-zero element of B is irregular (iff),*
2. *no non-zero element of B is regular (iff),*
3. *for some $b\in B\setminus\{0\}$, $bSb=\{0\}$ (iff),*
4. *for each $b\in B$, $bSb=\{0\}$*

[*in any of the above cases $B=\{b, 0\}$*];

5. *each element in B is regular (iff),*
6. *some non-zero element of B is regular (iff),*
7. *$bSb\neq\{0\}$ for each $b\in B\setminus\{0\}$ (iff),*
8. *$bSb\neq\{0\}$ for some $b\in B$*

[*in any of these cases B is a quasi-ideal*].

Proof. In any of the above cases one need consider only bSb for $b\in B$. We observe that bSb is a bi-ideal contained in B. Thus by the minimality of B either $bSb=\{0\}$ or $bSb=B$. In cases 1 or 2 if b is irregular then $b\notin bSb$ so $bSb\subset B$ and hence $bSb=\{0\}$. Clearly $\{b, 0\}$ is then a bi-ideal and hence $B=\{b, 0\}$. The equivalence of statements 1—4 should now be obvious.

Indeed, it is now clear that a non-zero element of B can be regular precisely when each element in B is regular. Furthermore $b\neq 0$ is regular iff $bRb\neq\{0\}$ since in such a case $bRb=B$. It follows that each of the statements 5—8 are equivalent and for any of these cases B is a quasi-ideal by (1. 9).

We give the analogous result for nilpotent minimal bi-ideals in rings.

(3. 4) Theorem. *Let B be a nilpotent minimal bi-ideal of a ring R. Then the following sets of equivalent statements are mutually exclusive.*

1. *some non-zero element of B is irregular (iff),*
2. *no non-zero element of B is regular (iff),*
3. *for some $b\in B\setminus\{0\}$, $bRb=\{0\}$ (iff),*

4. *for each $b \in B$, $bRb=\{0\}$*

[*in any of the above cases $B=([b], +)$ where b is of prime order*];

5. *each element in B is regular (iff),*
6. *some non-zero element of B is regular (iff),*
7. *$bRb \neq \{0\}$ for each $b \in B \setminus \{0\}$ (iff),*
8. *$bRb \neq \{0\}$ for some $b \in B$*

[*in any of these cases B is a quasi-ideal*].

Proof. The additive operation of R does not enter into consideration unti the final conclusion is approached. The proof of (3. 3) can be repeated intact. Now however if $bRb=\{0\}$, b will generate an additive subgroup $[b]$ which is a bi-ideal. Since $(nb)r(mb)=b(nmr)b=0$ any subgroup of $([b], +)$ will also be a bi-ideal. Thus $B=([b], +)$ and the order of b must clearly be finite (else take $[2b]$ etc.) and prime. If any of the conditions 5—8 hold B will be a quasi-ideal by (1. 10).

If B is a subgroup of a ring R, with $B^2=\{0\}$ and the order of B prime, then it is clear that if B is a bi-ideal it must be minimal. It suffices to have either R commutative or B contained in the center of R to have this be the case. As the following example shows it is possiblet o have a subgroup $(B, +)$ of prime order and $B^2=\{0\}$ without B being a bi-ideal.

(3. 5) Example. Let R be the ring of 4×4 matrices over $Z/(p)$, where Z is the ring of integers and p is a prime number. Let

$$B=\left\{\begin{pmatrix}0&0&0&0\\a&0&0&0\\0&0&0&0\\0&0&a&0\end{pmatrix}\middle|\quad a\in Z/(p)\right\}.$$

It is easy to check that $B^2=\{0\}$ but that B is not a bi-ideal of R. Moreover if we take here

$$S=\left\{\begin{pmatrix}0&0&0&0\\x&0&y&0\\0&0&0&0\\u&0&v&0\end{pmatrix}\middle|\quad x,y,u,v\in Z/(p)\right\}$$

then S is a bi-ideal of R and B a bi-ideal of S since $BS=\{0\}$. Thus the regularity condition of (1. 8) is in one sense necessary for the middle subring. Here R is, as is well known, a regular ring.

It is easy to observe from the above two theorems ((3. 3) and (3. 4)), (2. 4) and [20] Theorem 3 that if a minimal bi-ideal (or quasi-ideal) is either a division ring (or group union $\{0\}$ in the semigroup case) or nilpotent and possesses no non-zero regular element (regular in the original ring or semigroup) then the bi-ideal (or quasi-ideal) considered itself as a ring (or semigroup) contains no non-trivial bi-ideals (or quasi-ideals).

In the first case the bi-ideal is also a quasi-ideal. This situation is altered in the remaining case when the elements of the nilpotent minimal bi-ideal are regular (the second set of conditions in (3. 4) or (3. 5)). Here there may be many proper bi-ideals of the given minimal bi-ideal. We conclude with the following examples which illustrate this situation.

(3. 6) Example. Let S be a completely 0-simple semigroup over a non-trivial group, G, (cf. [1], [4]) where S is not a completely simple semigroup with a adjoined 0, i.e., S has at least one non-zero nilpotent $\mathcal{H}$-class. It is easy to see that the minimal bi-ideals of S are just individual non-zero $\mathcal{H}$-classes union $\{0\}$. Since S is regular these are also the minimal quasi-ideals of S (cf. [19], [22], [5]). Let B denote a non-zero nilpotent $\mathcal{H}$-class union $\{0\}$. Then B is a minimal bi-ideal satisfying the second set of conditions in (3. 3). Since G is non-trivial, $|B|>2$. It is easy to see since $B^2=\{0\}$ that any proper subset of B which contains 0 will be a bi-ideal (quasi-ideal) of B.

(3. 7) Example. Let Q denote the rational numbers and let R by the complete ring of 2×2 matrices over Q. R is a regular ring. Let $B=\left\{\begin{pmatrix}0 & 0\\ q & 0\end{pmatrix}\middle| q\in Q\right\}$. Then one readily checks that B is a nilpotent minimal bi-ideal (quasi-ideal) of R which falls under the second set of conditions in (3. 4). Again since $B^2=\{0\}$ any non-trivial subgroup (under addition) of B, and there are many, will be a bi-ideal (quasi-ideal) of B.

References

[1] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups.* Vol. 1, 2 (Providence, R. I., 1961 and 1967).

[2] R. A. Good and D. R. Hughes, Associated groups for a semigroup, *Bull. Amer. Math. Soc.*, **58** (1952), 624—625.

[3] K. Iseki, A characterization of regular semi-group, *Proc. Jap. Acad.*, **32** (1956), 676—677.

[4] K. M. Kapp and H. Schneider, *The lattice of congruences on a completely 0-simple semigroup*, Benjamin Press (New York, 1969).

[5] K. M. Kapp, Green's relation and quasi-ideals, *Czech. Math. J.*, **19 (94)** (1969), 80—85.

[6] ——— On bi-ideals and quasi-ideals in semigroups, *Publ. Math. Debrecen*, **16** (1969), 179—185.

[7] L. Kovács, A note on regular rings, *Publ. Math. Debrecen*, **4** (1956), 465—468.

[8] S. Lajos, A félcsoportok ideálelméletéhez, *Magyar Tud. Akad. Mat. Fiz. Oszt. Közl.*, **11** (1961), 57—61.

[9] ——— Generalized ideals in semigroups, *Acta Sci. Math.*, **22/23** (1961), 217—222.

[10] ——— On quasi-ideals of regular ring, *Proc. Japan Acad.*, **38** (1962), 210—211.

[11] ——— On the bi-ideals in semigroups, *Proc. Japan Acad.*, **45** (1969), 710—712.

[12] ——— Notes on (m, n)-ideals. I—III, *Proc. Japan Acad.*, **39** (1963), 419—421; **40** (1964), 631—632; **41** (1965), 383—385.

[13] S. Lajos and F. Szász, On the bi-ideals in associative rings, *Proc. Japan Acad.*, **46** (1970), 505—507.

[14] ——— *Bi-ideals in associative rings*, Dept. of Math., Karl Marx Univ. of Economics (Budapest, 1970.)

[15] Jiang Luh, A characterization of regular rings, *Proc. Japan Acad.*, **39** (1963), 741—742.

[16] ——— A remark on quasi-ideals of regular ring, *Proc. Japan Acad.*, **40** (1964), 660—661.

[17] O. Steinfeld, On ideal-quotients and prime ideals, *Acta Math. Sci. Hung.*, **4** (1953), 289—298.

[18] ——— Bemerkung zu einer Arbeit von T. Szele, *Acta Math. Acad. Sci. Hung.*, **6** (1955), 479—484.

[19] ——— Über die Quasi-ideale von Halbgruppen, *Publ. Math. Debrecen*, **4** (1965), 262—275.

[20] ——— Über die Quasi-ideale von Ringen, *Acta Sci. Math.*, **17** (1956), 170—180.

[21] ——— Über die Quasi-ideale von Halbgruppen mit eigentlichem Suschkewitsch-Kern, *Acta Sci. Math.*, **18** (1957), 235—242.

[22] ——— On semigroups which are unions of completely 0-simple semigroups, *Czech. Math. J.*, **16 (91)** (1966), 63—69.

[23] J. von Neumann, On regular rings, *Proc. Nat. Acad. Sci.*, **22** (1936), 707—713.

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