Independence of the conditions of associativity in ternary operations

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1. Let S be a set of elements $\{a, b, c, \ldots\}$. A ternary operation f is a mapping of $S \times S \times S$ to S. The operation f is said to be associative, if for every $a, b, c, d, e \in S$ we have

$$(*) \qquad [a, b, (c, d, e)_f]_f = [a, (b, c, d)_f, e]_f = [(a, b, c)_f, d, e]_f,$$

where $(x, y, z)_f$ denotes the image by f of the ordered triplet (x, y, z) in S. The equalities in (1. 1) are called *associativity conditions* for the elements a, b, c, d, e.

Associativity conditions in a set are said to be independent if any of them is not implied by the rest [1]. Thus, for ternary operations in S, the associativity conditions are independent if for every sequence $\{a, b, c, d, e\}$ of five elements in S it is possible to define a ternary operation f in S in such a way that the equalities $(*)$ hold for every sequence of elements $\{p, q, r, s, t\}$ of S different from $\{a, b, c, d, e\}$ whereas for this latter sequence $(*)$ does not hold.

2. G. SZÁSZ [2] has investigated the independence of associativity conditions for binary operations and has established the following theorem:

Theorem. *If the number of elements in a set S is greater than three, then the associativity conditions in S are independent.*

In what follows we study the same problem for ternary operations and prove:

Theorem. *If the number of elements in a set S is greater than five, then the associativity conditions for ternary operations in S are independent.*

3. Proof. For simplicity we drop the mapping letter f for the ternary operation.

Let $\{a, b, c, d, e\}$ be an arbitrary sequence of five elements of S. We shall prove the theorem by defining ternary operations over S such that the associativity conditions hold in all other cases except for this sequence. In what follows we consider separately the various alternatives for the elements of the sequence $\{a, b, c, d, e\}$ to prove the theorem.

3. 1. Let $a=b=c=d=e$. Since S contains more than five elements, we can choose three more distinct elements u, v, w different from a, and define the following operation in S:

$$(a, a, a)=u, \quad (a, a, u)=v, \quad \text{and} \quad (x, y, z)=w \quad \text{in all other cases.}$$

Clearly, the associativity conditions do not hold for the given sequence of elements, as

$$[(a, a, a), a, a]=(u, a, a)=w, \quad [a, (a, a, a), a]=(a, u, a)=w,$$

$$\text{but} \quad [a, a, (a, a, a)]=(a, a, u)=v.$$

We show that for every sequence $\{p, q, r, s, t\}$ different from $\{a, a, a, a, a\}$, the associativity conditions do hold.

Since (p, q, r) and (q, r, s) are never a, we have $[(p, q, r), s, t]=w$ and $[p, (q, r, s), t]=w$.

Now, if $(r, s, t)\neq u$, then as $(r, s, t)\neq a$, $[p, q, (r, s, t)]=w$. And if $(r, s, t)=u$ but $p\neq a$, even then $(p, q, u)=w$. And even if $(r, s, t)=u$, $p=a$, then $(a, q, u)=w$, except when $q=a$, in which case $\{p, q, r, s, t\}$ is the same as $\{a, a, a, a, a\}$, the given sequence. This proves the contention.

3. 2. Let $a\neq b$ and $b=c=d=e$ (all but one element being the same).

Since S contains more than five elements, we can choose an element w different from a and b, and define the following operation in S:

$$(a, b, b)=a \quad \text{and} \quad (x, y, z)=w \quad \text{in all other cases.}$$

We have for the given sequence $\{a, b, b, b, b\}$:

$$[a, b, (b, b, b)]=w \quad \text{and} \quad [a, (b, b, b), b]=w, \quad \text{but} \quad [(a, b, b), b, b]=(a, b, b)=a,$$

so that the associativity conditions do not hold in this case.

For any sequence $\{p, q, r, s, t\}$ different from $\{a, b, b, b, b\}$ we show that the associativity conditions do hold.

Since $(q, r, s)\neq b$ and $(r, s, t)\neq b$,

$$[p(q, r, s), t]=w \quad \text{and} \quad [p, q, (r, s, t)]=w.$$

Now, if $(p, q, r)\neq a$, then $[(p, q, r), s, t]=w$. And if $(p, q, r)=a$, but $s\neq b$, then $[(p, q, r), s, t]=(a, s, t)=w$. And even if $(p, q, r)=a$, $s=b$, then $[(p, q, r), s, t]=$ $=(a, b, t)=w$ except when $t=b$, in which case $\{p, q, r, s, t\}$ is the same as $\{a, b, b, b, b\}$, the given sequence.

3. 2.1. For another permutation of the same sequence of 3. 2 we prove the theorem as follows:

When the sequence is $\{b, a, b, b, b\}$, we can choose three more distinct elements u, v and w different from a and b (as S contains more than five elements) and then define the following operation in S:

$$(b, b, b)=u, \quad (b, a, u)=v, \quad \text{and} \quad (x, y, z)=w \quad \text{in all other cases.}$$

As $[(b, a, b), b, b]=w$, $[b, (a, b, b), b]=w$, and $[b, a, (b, b, b)]=(b, a, u)=v$, we see that the associativity conditions do not hold for the given sequence $\{b, a, b, b, b\}$.

For any sequence $\{p, q, r, s, t\}$ different from $\{b, a, b, b, b\}$ we show that the conditions do hold.

Since (p, q, r) and (q, r, s) can never be equal to a or b,

$$[(p, q, r), s, t]=w \quad \text{and} \quad [p, (q, r, s), t]=w.$$

Now, if $(r, s, t)\neq u$, then $[p, q, (r, s, t)]=w$, as $(r, s, t)\neq b$. And, if $(r, s, t)=u$, but $p\neq b$, then $[p, q, (r, s, t)]=w$. And even if $(r, s, t)=u$, $p=b$, $[p, q, (r, s, t)] = (b, q, u) = w$, except when $q=a$, in which case $\{p, q, r, s, t\}$ is the same as $\{b, a, b, b, b\}$, the given sequence.

By defining a similar operation in S, we can demonstrate the truth of the theorem in the same way for every other sequence of five elements in which four elements are equal but the fifth is different*).

3. 3. Let $a=b$, $c=d=e$ and $a\neq c$.

Since S contains more than five elements we can choose three more distinct elements u, v, and w different from a and c and define the operation as follows:

$$(c, c, c)=u, \quad (a, a, u)=v, \quad \text{and} \quad (x, y, z)=w \quad \text{in all other cases.}$$

We have for the given sequence

$$[(a, a, c), c, c]=w \quad \text{and} \quad [a, (a, c, c), c]=w, \quad \text{but} \quad [a, a, (c, c, c)]=(a, a, u)=v,$$

so that the associativity conditions do not hold, whereas we show that for any sequence $\{p, q, r, s, t\}$ different from $\{a, a, c, c, c\}$ the conditions do hold.

Since (p, q, r) and (q, r, s) can never be equal to a or c, we get

$$[(p, q, r), s, t]=w \quad \text{and} \quad [p, (q, r, s), t]=w.$$

Now if $(r, s, t)\neq u$, then $[p, q, (r, s, t)]=w$ as $(r, s, t)\neq c$. Also, if $(r, s, t)=u$, but $p\neq a$, $[p, q, (r, s, t)]=(p, q, u)=w$, and even if $(r, s, t)=u$ and $p=a$, $[p, q, (r, s, t)]=$

*) The authors have investigated each of these permutations also, but for brevity all these have not been incorporated here.

$=(a, q, u)=w$ except when $q=a$, in which case $\{p, q, r, s, t\}$ is the same as the given sequence $\{a, a, c, c, c\}$.

We can similarly prove the statement of the theorem for other arrangements of the elements in the sequence $\{a, a, c, c, c\}$ such as $\{a, c, a, c, c\}$, $\{a, c, c, a, c\}$ etc. by defining similar operations.

3. 4. Let $a=b$, $c=d$ and let e be different from a and c.

Since, under the hypothesis, S has more than five elements, we can choose three more distinct elements u, v, and w different from a, c, and e and define the following operation:

$$(c, c, e)=u, \quad (a, a, u)=v, \quad \text{and} \quad (x, y, z)=w \quad \text{in all other cases.}$$

We have for the given sequence

$$[(a, a, c), c, e]=w, \quad [a, (a, c, c,), e]=w, \quad \text{but} \quad [a, a, (c, c, e)]=(a, a, u)=v,$$

so that the associativity conditions do not hold, whereas we show that for any sequence $\{p, q, r, s, t\}$ different from $\{a, a, c, c, e\}$ the conditions do hold.

Since (p, q, r) and (q, r, s) can never be equal to a or c, we have $[(p, q, r), s, t]=w$ and $[p, (q, r, s), t]=w$. Now if $(r, s, t)\neq u$, then $[p, q, (r, s, t)]=w$ as $(r, s, t)\neq e$. Also, if $(r, s, t)=u$, but $p\neq a$, then $[p, q, (r, s, t)]=w$. And even if $(r, s, t)=u$, $p=a$, then $[p, q, (r, s, t)]=w$ except when $q=a$, in which case $\{p, q, r, s, t\}$ is the same as the given sequence $\{a, a, c, c, e\}$.

We can have a similar proof for other arrangements of the elements in the sequence $\{a, a, c, c, e\}$.

3. 5. Let $a=b=c$, $d\neq e$, and let d, e be different from a.

Since S contains more than five elements, we can choose three distinct elements u, v, w different from a, d, e and then define the following operation:

$$(a, d, e)=u, \ (a, a, u)=v, \quad \text{and} \quad (x, y, z)=w \quad \text{in all other cases.}$$

Here we see that the associativity conditions do not hold for the given sequence. For,

$$[(a, a, a), d, e]=w, \ [a, (a, a, d), e]=w, \quad \text{but} \quad [a, a, (a, d, e)]=(a, a, u)=v.$$

For any sequence $\{p, q, r, s, t\}$ different from $\{a, a, a, d, e\}$, we show that the conditions do hold.

Since (p, q, r) can never be equal to a and (q, r, s) can never be equal to a or d, we have

$$[(p, q, r)s, t]=w \quad \text{and} \quad [p, (q, r, s), t]=w.$$

Now if $(r, s, t)\neq u$, then $[p, q, (r, s, t)]=w$ as $(r, s, t)\neq e$. Also if $(r, s, t)=u$, but

$p \neq a$, then $[p, q, (r, s, t)] = w$. And even if $(r, s, t) = u$, $p = a$, then $[p, q, (r, s, t)] = (a, q, u) = w$, except when $q = a$, in which case $\{p, q, r, s, t\}$ is the same as the given sequence $\{a, a, a, d, e\}$.

We can have a similar proof for other arrangements of the elements in the sequence $\{a, a, a, d, e\}$.

3. 6. Let $a = c$, and let b, d, e be all distinct and different from a.

Since S contains more than five elements we can choose two distinct elements u and w in S, different from a, b, d, and e, and define the operation as follows:

$$(a, b, a) = a, \quad (b, a, b) = b, \quad (a, d, e) = u,$$

and

$$(x, y, z) = w \quad \text{in all other cases.}$$

We see that the associativity conditions do not hold for the given sequence, for

$$[a, b, (a, d, e)] = (a, b, u) = w, \quad [a, (b, a, d), e] = (a, w, e) = w,$$

but

$$[(a, b, a), d, e] = (a, d, e) = u.$$

For any sequence $\{p, q, r, s, t\}$ different from $\{a, b, a, d, e\}$ and containing an element different from a, b, d, e, the conditions do hold, as in all such cases

$$[(p, q, r), s, t] = w, \quad [p, (q, r, s), t] = w, \quad [p, q, (r, s, t)] = w.$$

Hence taking p, q, r, s, t all from a, b, d, e only, we show that the associativity conditions do hold in each case. We distinguish five cases, denoted by the letters (A)—(E).

(A) When $p \neq a$, we have $[(p, q, r), s, t] = (x, s, t)$, say, where $x = b$ or $x = w$.

(i) Let $x = b$ i.e. $(p, q, r) = (b, a, b)$. Then $[(b, a, b), s, t] = (b, s, t) = b$ when $s = a$, $t = b$ in which case

$$[b, (a, b, a), b] = (b, a, b) = b \quad \text{and} \quad [b, a, (b, a, b)] = (b, a, b) = b.$$

And when $s \neq a$ or $t \neq b$,

$$[(b, a, b)s, t] = [b, (a, b, s), t] = [b, a, (b, s, t)] = w.$$

(ii) Let $x = w$. Then

$$[(p, q, r), s, t] = (w, s, t) = w;$$

and

$$[p, (q, r, s), t] = (p, a, t) \quad \text{or} \quad (p, b, t), \quad \text{or} \quad (p, u, t), \quad \text{or} \quad (p, w, t).$$

If $(q, r, s)=a$ i.e. $(q, r, s)=(a, b, a)$, p cannot be equal to b, for otherwise $(p, q, r)=b$. Then

$$(p, a, t)=(p, b, t)=(p, u, t)=(p, w, t)=w$$

and $[p, q, (r, s, t)]=(p, q, a)$ or (p, q, b) or (p, q, u) or (p, q, w).

Now, if $(r, s, t)=b$, p and q can never have values b and a, respectively, for $(p, q, r)= =w$. Therefore

$$(p, q, a)=(p, q, b)=(p, q, u)=(p, q, w)=w,$$

Hence

$$[(p, q, r), s, t]=[p, (q, r, s), t]=[p, q, (r, s, t)]=w.$$

(B) When $p=a$ but $q\neq b$, we have

$$[(a, q, r), s, t]=[a, (q, r, s), t]=[a, q, (r, s, t)]=w,$$

for $(a, q, r)\neq a$ or b, $(q, r, s)\neq b$ or d, $(r, s, t)\neq e$.

(C) When $p=a$, $q=b$ but $r\neq a$, we have

$$[(a, b, r), s, t]=[a, (b, r, s), t]=[a, b, (r, s, t)]=w.$$

(D) When $p=a$, $q=b$, $r=a$ but $s\neq d$, we have

$$[(p, q, r), s, t]=[(a, b, a), s, t]=(a, s, t)=a \quad (\text{when } s=b,\ t=a)$$

in which case

$$[a, (b, a, b), a]=(a, b, a)=a \text{ and } [a, b, (a, b, a)]=(a, b, a)=a, \text{ and (when } s\neq b)$$

in all other cases

$$[(a, b, a), s, t]=[a, (b, a, s), t]=[a, b, (a, s, t)]=w.$$

(E) When $p=a$, $q=b$, $r=a$, $s=d$ but $t\neq e$, we have

$$[(a, b, a), d, t]=[a, (b, a, d), t]=[a, b, (a, d, t])=w.$$

This proves the contention.

3. 7. Let a, b, c, d, e be all different from one another.

As S contains more than five elements, we can choose an element w different from the given five elements and then define the operation in S as follows:

$$(c, d, c)=c, \quad (a, b, e)=a, \quad (d, e, b)=d,$$
$$(d, c, d)=d, \quad (b, e, b)=b, \quad (e, b, c)=c,$$
$$(c, d, e)=e, \quad (e, b, e)=e, \quad (b, c, d)=b, \text{ and}$$
$$(x, y, z)=w \text{ in all other cases.}$$

For the given sequence

$$[a, b, (c, d, e)]=(a, b, e)=a, \quad [a, (b, c, d), e]=(a, b, e)=a,$$

but

$$[(a, b, c), d, e]=(w, d, e)=w$$

so that the associativity conditions do not hold. For any sequence $\{p, q, r, s, t\}$ different from $\{a, b, c, d, e\}$ and containing an element different from a, b, c, d and e, we have

$$[(p, q, r), s, t]=[p, (q, r, s), t]=[p, q, (r, s, t)]=w.$$

Hence taking p, q, r, s, t all from a, b, c, d, e only, we demonstrate that the associativity conditions do hold in each case as follows:

(A) When $p\neq a$, $[(p, q, r), s, t]=(x, s, t)$, say, then $x\neq a$.

(i) Let $x=b$ i.e. $(p, q, r)=(b, e, b)$ or $(p, q, r)=(b, c, d)$. Then in the first case

$$[(p, q, r), s, t]=[(b, e, b), s, t]=(b, s, t)=b \text{ (when } s=e,\ t=b \text{ or } s=c,\ t=d),$$

and hence

$$[b, (e, b, e), b]=(b, e, b)=b \quad \text{and} \quad [b, e, (b, e, b)]=(b, e, b)=b,$$

$$[b, (e, b, c), d]=(b, c, d)=b \quad \text{and} \quad [b, e, (b, c, d)]=(b, e, b)=b,$$

while in the second case

$$[(p, q, r), s, t]=[(b, c, d), s, t]=(b, s, t)=b \text{ (when } s=e,\ t=b, \text{ or } s=c,\ t=d)$$

and hence

$$[b, (c, d, e), b]=(b, e, b)=b \quad \text{and} \quad [b, c, (d, e, b)]=(b, c, d)=b,$$

$$[b, (c, d, c), d]=(b, c, d)=b \quad \text{and} \quad [b, c, (d, c, d)]=(b, c, d)=b,$$

and $[(p, q, r), s, t]=[p, (q, r, s), t]=[p, q, (r, s, t)]=w$ in other cases.

(ii) Let $x=c$, i.e. $(p, q, r)=(c, d, c)$ or $(p, q, r)=(e, b, c)$.

Then

$$[(p, q, r), s, t]=[(c, d, c), s, t]=(c, s, t) = \begin{cases} c, & \text{if } s=d,\ t=c, \\ e, & \text{if } s=d,\ t=e; \end{cases}$$

in these cases

$$[c, (d, c, d), c]=(c, d, c)=c \quad \text{and} \quad [c, d, (c, d, c)]=(c, d, c)=c, \quad \text{and}$$

$$[c, (d, c, d), e]=(c, d, e)=e \quad \text{and} \quad [c, d, (c, d, e)]=(c, d, e)=e,$$

and $[(p, q, r), s, t]=[(e, b, c), s, t]=(c, s, t)=c$ (when $s=d$ and $t=c$) or $=e$ (when $s=d$ and $t=e$), in which case

$$[e, (b, c, d), c]=(e, b, c)=c \quad \text{and} \quad [e, b, (c, d, c)]=(e, b, c)=c,$$

$$[e, (b, c, d), e]=(e, b, e)=e \quad \text{and} \quad [e, b, (c, d, e)]=(e, b, e)=e,$$

and $[(p, q, r), s, t]=[p, (q, r, s), t]=[p, q, (r, s, t)]=w$ in other cases.

(iii) Let $x=d$, i.e. $(p, q, r)=(d, c, d)$ or $(p, q, r)=(d, e, b)$. Then $[(p, q, r), s, t]=$ $=(d, s, t)=d$ (when $s=c$, $t=d$, or, when $s=e$, $t=b$),

in which case

$$[d, (c, d, c), d]=(d, c, d)=d \quad \text{and} \quad [d, c, (d, c, d)]=(d, c, d)=d,$$

$$[d, (c, d, e), b]=(d, e, b)=d \quad \text{and} \quad [d, c, (d, e, b)]=(d, c, d)=d,$$

$$[d, (e, b, c), d]=(d, c, d)=d \quad \text{and} \quad [d, e, (b, c, d)]=(d, e, b)=d,$$

$$[d, (e, b, e), b]=(d, e, b)=d \quad \text{and} \quad [d, e, (b, e, b)]=(d, e, b)=d,$$

and $[(p, q, r), s, t]=[p, (q, r, s), t]=[p, q, (r, s, t)]=w$ in all the other cases.

(iv) Let $x=e$, i.e. $(p, q, r)=(c, d, e)$ or $(p, q, r)=(e, b, e)$.

Then

$$[(p, q, r), s, t]=(e, s, t) = \begin{cases} c, & \text{if } s=b, \ t=c, \\ e, & \text{if } s=b, \ t=e, \end{cases}$$

and in each case

$$[c, (d, e, b), c]=(c, d, c)=c \quad \text{and} \quad [c, d, (e, b, c)]=(c, d, c)=c,$$

$$[c, (d, e, b), e]=(c, d, e)=e \quad \text{and} \quad [c, d, (e, b, e)]=(c, d, e)=e,$$

$$[e, (b, e, b), c]=(e, b, c)=c \quad \text{and} \quad [e, b, (e, b, c)]=(e, b, c)=c,$$

$$[e, (b, e, b), e]=(e, b, e)=e \quad \text{and} \quad [e, b, (e, b, e)]=(e, b, e)=e,$$

and $[(p, q, r,)s, t]=[p, (q, r, s), t]=[p, q, (r, s, t)]=w$ in all other cases.

(v) Let $x=w$, i.e. $(p, q, r)=w$. Then $[(p, q, r), s, t]=(w, s, t)=w$. Furthermore, $[p, (q, r, s), t]=w$, for

$$(p, a, t)=w \text{ for all values of } p \text{ and } t,$$

$$(p, b, t)=w \text{ as } p\neq a \text{ and } p\neq e,$$

$$(p, c, t)=w \text{ as } p\neq b \text{ and } p\neq d,$$

$$(p, d, t)=w \text{ as } p\neq c,$$

$$(p, e, t)=w \text{ as } p\neq b \text{ and } p\neq d,$$

$$(p, w, t)=w \text{ for all values of } p \text{ and } t.$$

Again, in each case $[p, q, (r, s, t)]=w$, because

$$(p, q, a)=w \text{ for all values of } p \text{ and } q,$$

$$(p, q, b)=w \text{ as } (p\neq b, q\neq e) \text{ and } (p\neq d, q\neq e),$$

$$(p, q, c)=w \text{ as } (p\neq c, q\neq d) \text{ and } (p\neq e, q\neq b),$$

$$(p, q, w)=w \text{ for all values of } p \text{ and } q,$$

$$(p, q, d)=w \text{ as } (p\neq d, q\neq c) \text{ and } (p\neq b, q\neq c),$$

$$(p, q, e)=w \text{ as } (p\neq e, q\neq b) \text{ and } (p\neq a, q\neq b), (p\neq c, q\neq d).$$

(B) When $p=a$, but $q\neq b$, then

$$[(a, q, r), s, t]=(w, s, t)=w \quad \text{and} \quad [a, (q, r, s), t]=w, [a, q, (r, s, t)]=w.$$

(C) When $p=a$, $q=b$, but $r\neq c$, then

$$[(p, q, r), s, t]=[(a, b, r), s, t]=(a, s, t)=a \quad \text{if} \quad r=e,\ s=b,\ t=e,$$

in which case

$$[a, (b, e, b), e]=(a, b, e)=a \quad \text{and} \quad [a, b, (e, b, e)]=(a, b, e)=a,$$

and $[(a, b, r), s, t]=[a, (b, r, s), t]=[a, b, (r, s, t)]=w$ in all other cases.

(D) When $p=a$, $q=b$, $r=c$, but $s\neq d$, then

$$[(a, b, c), s, t]=[a, (b, c, s), t]=[a, b, (c, s, t)]=w.$$

(E) When $p=a$, $q=b$, $r=c$, $s=d$, but $t\neq e$, then

$$[(a, b, c), d, t]=[a, (b, c, d), t]=[a, b, (c, d, t)]=w.$$

Combining 3. 1 and 3. 7 together completes the proof of the theorem.

References

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