

On a certain converse of Hölder's inequality. II

By L. LEINDLER in Szeged

A. PRÉKOPA [2] proved the integral inequality

$$(1) \quad \int_{-\infty}^{\infty} \sup_{x+y=t} f(x)g(y) dt \cong 2 \left[\int_{-\infty}^{\infty} f^2(x) dx \right]^{1/2} \left[\int_{-\infty}^{\infty} g^2(y) dy \right]^{1/2},$$

for arbitrary Lebesgue measurable non-negative functions $f(x)$ and $g(y)$.

In [1] we proved the inequality of similar type

$$(2) \quad \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} (f(x)g(t-x))^{\gamma} dx \right]^{1/\gamma} dt \cong \left[\int_{-\infty}^{\infty} f^r(x) dx \right]^{1/r} \left[\int_{-\infty}^{\infty} g^s(x) dx \right]^{1/s},$$

where $1 \leq r, s, \gamma \leq \infty$ and $\frac{1}{r} + \frac{1}{s} = 1 + \frac{1}{\gamma}$. The proof of (2) is much easier than that of (1), but (2) does not include (1) because of the lack of the factor 2.

In the present paper we prove the inequality, more general than (1),

$$(3) \quad \int_{-\infty}^{\infty} \sup_{x+y=t} f(x)g(y) dt \cong p^{1/p} q^{1/q} \left[\int_{-\infty}^{\infty} f^p(x) dx \right]^{1/p} \left[\int_{-\infty}^{\infty} g^q(x) dx \right]^{1/q},$$

where $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Here the constant factor at the second member is best possible (if $p=1$ then by this constant we understand 1).

This inequality can be generalized to any finite number of functions as follows:

Theorem. Suppose that $1 \leq p_i \leq \infty$ ($i=1, 2, \dots, n$) and $\sum_{i=1}^n \frac{1}{p_i} = 1$. Then for arbitrary non-negative Lebesgue measurable functions $f_1(x^1), f_2(x^2), \dots, f_n(x^n)$ we have:

$$(4) \quad \int_{-\infty}^{\infty} \left[\sup_{\sum_{i=1}^n \frac{x^i}{p_i} = t} \prod_{i=1}^n f_i(x^i) \right] dt \cong \prod_{i=1}^n \|f_i\|_{p_i}$$

and this inequality is best possible.

For $n=2$ inequality (4) reduces to (3) by the substitution $x^1=px$ and $x^2=qy$. If $n=1$, then (4) reduces to the obvious equality:

$$(5) \quad \int_{-\infty}^{\infty} \sup_{x=t} f_1(x) dt = \int_{-\infty}^{\infty} f_1(x) dx = \|f_1\|_1.$$

We also obtain (5) if n is arbitrary, but one of the numbers p_i , say p_1 , is equal to 1; for in this case $p_2=p_3=\dots=p_n=\infty$, and thus we can divide (4) by $\prod_{i=2}^n \|f_i\|_{\infty}$ assuming this product is positive and finite (otherwise both sides of (4) are zero or infinite).

For similar reason, if one of the numbers p_i , say p_n , is equal to infinity, then we can divide (4) by $\|f_n\|_{\infty}$.

Therefore we may assume that $1 < p_i < \infty$ for all i .

Proof of the Theorem. We may assume, as already explained, that $1 < p_i < \infty$ for $i=1, 2, \dots, n$; for $n \geq 2$, the integral on the left-hand side of (4) has finite value, and the functions $f_i(x^i)$ do not vanish almost everywhere. We prove (4) for step functions with integer points of discontinuity only, the transition to arbitrary Lebesgue measurable functions follows as in [2]. Moreover, it suffices to consider step functions which at their points of discontinuity are equal to the larger one of the values taken on the adjoining intervals (this latter convention will be important technically later, see (10), (11) and (12)).

First we set down some notations and definitions: Let

$$F_i(x^i) = \frac{1}{\max f_i} f_i(x^i) \quad (i = 1, 2, \dots, n),$$

and N be an integer such that if $|x^i| > N$ then $f_i(x^i) = 0$ for all i ; furthermore let

$$F_i(x^i) = a_k^i \quad \text{if } x^i \in (k-1, k), \quad k = -N+1, -N+2, \dots, N-1, N.$$

Let v_i denote a fixed index with $a_{v_i}^i = 1$. Finally we define the following auxiliary function:

$$G_i(x^i) = \begin{cases} F_i(x^i) & \text{if } x^i \notin (v_i-1, v_i], \\ 2^{1/p_i} & \text{if } x^i \in [v_i-1, v_i]. \end{cases}$$

Denoting the values of $G_i(x^i)$ on $(k-1, k)$ by b_k^i , it is clear that $b_k^i = a_k^i$ if $k \neq v_i$ and for $k = v_i$ $b_{v_i}^i = 2^{1/p_i}$.

By means of these functions $G_i(x^i)$ we want to give a decomposition of the

interval $(-\infty < t < \infty)$ such that the sum of the lower estimations to be given on the subintervals for the left-hand side of (4) be greater than

$$\left[\prod_{i=1}^n (\max f_i) \right] \sum_{i=1}^n \frac{1}{p_i} \int_{-\infty}^{\infty} F_i^{p_i}(x^i) dx^i.$$

From this point the proof of (4) will already be easy.

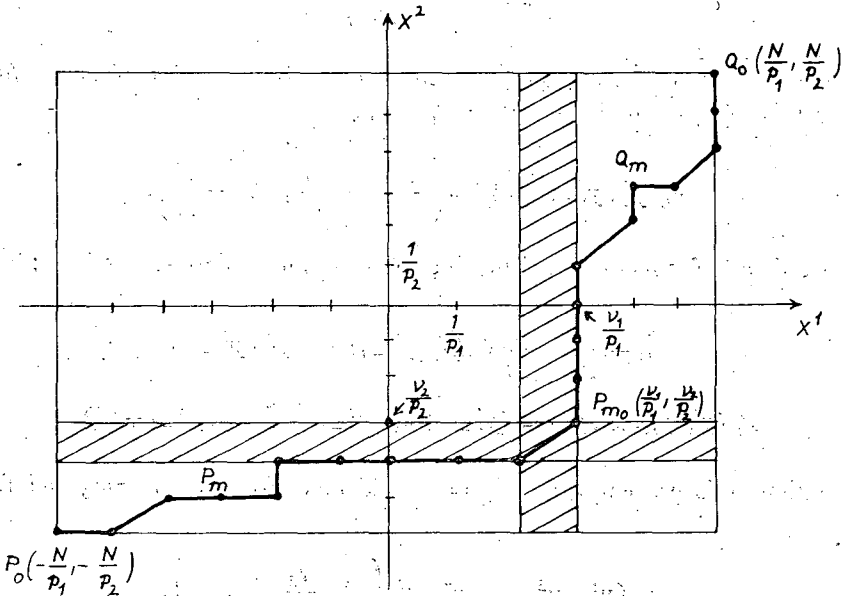
By the definition of N we have

$$(6) \quad S \equiv \int_{-\infty}^{\infty} \left[\sup_{\sum_{i=1}^n \frac{x^i}{p_i} = t} \prod_{i=1}^n F_i(x^i) \right] dt = \int_{-N}^N \left[\sup_{\sum_{i=1}^n \frac{x^i}{p_i} = t} \prod_{i=1}^n F_i(x^i) \right] dt \equiv S_N,$$

thus it is enough to decompose the interval $[-N, N]$.

First we sketch the idea of the decomposition in the case $n=2$.

We want to choose a path from the point $P_0 \left(-\frac{N}{p_1}, -\frac{N}{p_2} \right)$ to the point $Q_0 \left(\frac{N}{p_1}, \frac{N}{p_2} \right)$ such that by means of the "break points" of this path the required decomposition of the interval $[-N < t < N]$ could be given. See the following figure:



From a break point, e.g. from the point $P_m \left(\frac{k-1}{p_1}, -\frac{1}{p_2} \right)$ we go a step to the right or upward according as $(b_i^2)^{p_2}$ or $(b_i^1)^{p_1}$ is the larger number, that is, we go toward

the direction where the smaller value of the functions $(G_1(x^1))^{p_1}$ and $(G_2(x^2))^{p_2}$ in the following step can be found; if $(b_l^1)^{p_2} = (b_k^1)^{p_1}$ then we go obliquely upward to the point $\left(\frac{k}{p_1}, \frac{l}{p_2}\right)$. We continue this procedure till one of the break points P_m , say $m = m_0$, comes up to the point $\left(\frac{v_1}{p_1}, \frac{v_2}{p_2}\right)$. This ensues necessarily since one of the break points "knocks against the lined wall" and after this the points go along the wall to the point $\left(\frac{v_1-1}{p_1}, \frac{v_2-1}{p_2}\right)$ (in fact, the functions take the largest value on the wall), and from the point $\left(\frac{v_1-1}{p_1}, \frac{v_2-1}{p_2}\right)$, by $(b_{v_1}^1)^{p_1} = (b_{v_2}^2)^{p_2} = 2$, we jump to the point $\left(\frac{v_1}{p_1}, \frac{v_2}{p_2}\right)$. For similar reasons and by an analogous method we can come back from the point Q_0 to the point $\left(\frac{v_1}{p_1}, \frac{v_2}{p_2}\right)$ along the points Q_m . If we join the points Q_m to the points P_m in reverse order as we obtained them, then we will get the required path from P_0 to Q_0 .

Now we construct such a path in the n -dimensional case. Let

$$s(t) = \begin{cases} 1, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0, \end{cases}$$

and we denote, as usual, by $h_+(u_0)$ the limit from the right of the function $h(u)$ at u_0 , and by $h_-(u_0)$ the limit from the left. We put

$$P_0(y_0^1, y_0^2, \dots, y_0^n) = \left(-\frac{N}{p_1}, -\frac{N}{p_2}, \dots, -\frac{N}{p_n}\right).$$

Next we define, for $m \geq 1$, the following numbers and points successively:

$$u_m^i = \frac{1}{p_i} s(\min_{j \neq i} G_{j+}^{p_j}(p_j y_{m-1}^j) - G_{i+}^{p_i}(p_i y_{m-1}^i))$$

and

$$P_m(y_m^1, y_m^2, \dots, y_m^n) = (y_{m-1}^1 + u_m^1, y_{m-1}^2 + u_m^2, \dots, y_{m-1}^n + u_m^n).$$

We continue this procedure till $y_{m_0}^i = \frac{v_i}{p_i}$ will hold, for some $m = m_0$ and for all i , i.e.

$$P_{m_0}(y_{m_0}^1, y_{m_0}^2, \dots, y_{m_0}^n) \equiv \left(\frac{v_1}{p_1}, \frac{v_2}{p_2}, \dots, \frac{v_n}{p_n}\right).$$

This follows necessarily by the same reason as we explained it in the case of two functions.

Then we define a sequence of points $Q_m(z_m^1, z_m^2, \dots, z_m^n)$ in an analogous way coming back from the point $\left(\frac{N}{p_1}, \frac{N}{p_2}, \dots, \frac{N}{p_n}\right)$. Let

$$Q_0(z_0^1, z_0^2, \dots, z_0^n) = \left(\frac{N}{p_1}, \frac{N}{p_2}, \dots, \frac{N}{p_n}\right).$$

Similarly as before, we define the following numbers and points, for $m \geq 1$, successively:

$$v_m^i = \frac{1}{p_i} s(\min_{j \neq i} G_{j-}^{p_j} (p_j z_{m-1}^j) - G_{i-}^{p_i} (p_i z_{m-1}^i))$$

and

$$Q_m(z_m^1, z_m^2, \dots, z_m^n) = (z_{m-1}^1 - v_m^1, z_{m-1}^2 - v_m^2, \dots, z_{m-1}^n - v_m^n).$$

In the n -dimensional case we also "knock against the wall", therefore it is clear that in a finite number of steps, say in m_1 , we come to the point P_{m_0} , i.e. $P_{m_0} = Q_{m_1}$. For each i ($i=1, 2, \dots, n$) we put

$$y_{m_0+l}^i = z_{m_1-l}^i \quad (l=0, 1, \dots, m_1),$$

hereby we arranged the points in a sequence $P_m(y_m^1, y_m^2, \dots, y_m^n)$ ($m=0, 1, \dots, m_0+m_1$), which gives the required path from P_0 to Q_0 .

Now we prove that by means of the obtained path, i.e. by means of the sequence $P_m(y_m^1, y_m^2, \dots, y_m^n)$ ($m=0, 1, \dots, m_0+m_1$), we can give the decomposition of the interval $[-N, N]$ we have required. First we set for each i ($i=1, 2, \dots, n$)

$$(7) \quad I_m^i = y_m^i - y_{m-1}^i, \quad (m = 1, 2, \dots, m_0+m_1),$$

furthermore denote by c_m^i the value of $F_i(x^i)$ on the interval $(p_i y_{m-1}^i, p_i y_m^i)$ if $I_m^i = \frac{1}{p_i}$, and at the point $x^i = p_i y_m^i$ if $I_m^i = 0$.

Let

$$(8) \quad t_k = \sum_{i=1}^n y_k^i \quad (k = 0, 1, \dots, m_0+m_1).$$

It is easy to see that $t_0 = -N$ and $t_{m_0+m_1} = N$, furthermore for $k \geq 1$

$$t_k = t_{k-1} + t_k - t_{k-1} = t_{k-1} + \sum_{i=1}^n I_k^i$$

also holds. Thus we can decompose each interval $[t_{k-1}, t_k]$ by the points

$$(9) \quad \tau_{k,0} = t_{k-1} \quad \text{and} \quad \tau_{k,j} = t_{k-1} + \sum_{i=1}^j I_k^i \quad (j = 1, 2, \dots, n).$$

On such an interval $[\tau_{k,j-1}, \tau_{k,j}]$ for any k and j ($k=1, 2, \dots, m_0+m_1; j=1, 2, \dots, n$) we have the following lower estimate:

$$(10) \quad S_{k,j} \equiv \int_{\tau_{k,j-1}}^{\tau_{k,j}} \left[\sup_{\sum_{i=1}^n \frac{x^i}{p_i} = t} \prod_{i=1}^n F_i(x^i) \right] dt \geq I_k^j \prod_{i=1}^n c_k^i.$$

To verify (10) we put $x^i = y_k^i p_i$ for $i < j$ and $x^i = y_{k-1}^i p_i$ for $i > j$, and we have x^j run from $y_{k-1}^j p_j$ to $y_k^j p_j$, then t goes from $\tau_{k,j-1}$ to $\tau_{k,j}$; in fact, by (7), (8) and (9) we have

$$t = \sum_{i=1}^n \frac{x^i}{p_i} \geq \sum_{i=1}^{j-1} y_k^i + \sum_{i=j}^n y_{k-1}^i = t_{k-1} + \sum_{i=1}^{j-1} I_k^i = \tau_{k,j-1}$$

and

$$t = \sum_{i=1}^n \frac{x^i}{p_i} \leq \sum_{i=1}^j y_k^i + \sum_{i=j+1}^n y_{k-1}^i = t_{k-1} + \sum_{i=1}^j I_k^i = \tau_{k,j}.$$

Choosing x^i in such a way as we mentioned above, we have

$$(11) \quad S_{k,j} = \int_{\tau_{k,j-1}}^{\tau_{k,j}} \left[\sup_{\sum_{i=1}^n \frac{x^i}{p_i} = t} \prod_{i=1}^n F_i(x^i) \right] dt \geq \\ \geq \prod_{i < j} F_i(y_k^i p_i) \prod_{i > j} F_i(y_{k-1}^i p_i) c_k^j \int_{\tau_{k,j-1}}^{\tau_{k,j}} dt,$$

and hence, by the definition of our step functions (see their definition at the points of discontinuity), (10) obviously follows.

By (9) and (10) we obtain

$$(12) \quad \sigma_k = \sum_{j=1}^n S_{k,j} = \int_{\tau_{k-1}}^{\tau_k} \left[\sup_{\sum_{i=1}^n \frac{x^i}{p_i} = t} \prod_{i=1}^n F_i(x^i) \right] dt \geq \left(\sum_{i=1}^n I_k^i \right) \prod_{i=1}^n c_k^i.$$

By the definitions of y_k^i and I_k^i , furthermore taking into account that the functions at their points of discontinuity are equal to the larger one of the values taken on the adjoining intervals, it can be seen that I_k^i differs from zero only for such indices j for which $(c_k^j)^{p_j} \equiv (c_k^i)^{p_i}$ holds for all i ($i=1, 2, \dots$). Thus we obtain from (12) that

$$(13) \quad \sigma_k \geq \sum_{j=1}^n I_k^j (c_k^j)^{p_j},$$

since if $I_k^i \neq 0$ then

$$\prod_{i=1}^n c_k^i \geq \prod_{i=1}^n (c_k^i)^{\frac{p_j}{p_i}} = (c_k^j)^{\sum_{i=1}^n \frac{p_j}{p_i}} = (c_k^j)^{p_j},$$

whence, by (12), inequality (13) follows obviously.

Since

$$S = S_N = \sum_{k=1}^{m_0+m_1} \sigma_k,$$

by (13), we have

$$(14) \quad S \cong \sum_{k=1}^{m_0+m_1} \sum_{j=1}^n I_k^j (c_k^j)^{p_j} = \sum_{j=1}^n \sum_{k=1}^{m_0+m_1} I_k^j (c_k^j)^{p_j}.$$

By the definition of I_k^j and c_k^j it is clear that

$$(15) \quad \sum_{k=1}^{m_0+m_1} I_k^j (c_k^j)^{p_j} = \frac{1}{p_j} \sum_{l=-N+1}^N (a_l)^{p_j},$$

thus, by (14) and (15), using the following well-known inequality $\prod_{i=1}^n \varrho_i \cong \sum_{i=1}^n \frac{1}{p_i} (\varrho_i)^{p_i}$ ($\varrho_i > 0$), we get

$$(16) \quad S \cong \sum_{j=1}^n \frac{1}{p_j} \sum_{l=-N+1}^N (a_l)^{p_j} \cong \prod_{j=1}^n \left(\sum_{l=-N+1}^N (a_l)^{p_j} \right)^{1/p_j} = \prod_{j=1}^n \|F_j\|_{p_j}.$$

Multiplying both sides of (16) by $\prod_{i=1}^n (\max f_i)$ we obtain the required inequality (4).

To prove that the inequality (4) is best possible we define the following functions:

$$f_i^0(x^i) = \begin{cases} 1 & \text{on } (0, 1), \\ 0 & \text{otherwise,} \end{cases} \quad i = 1, 2, \dots, n.$$

Then

$$\int_{-\infty}^{\infty} \left[\sup_{\sum_{i=1}^n \frac{x^i}{p_i} = t} \prod_{i=1}^n f_i^0(x^i) \right] dt = \int_0^1 1 dt = \prod_{i=1}^n \|f_i^0\|_{p_i}.$$

The proof is thus completed.

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