Compact restrictions of operators. II

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1. Introduction. As the title indicates, this paper is a continuation of [1], and, accordingly, we shall assume that the reader is familiar with the results and terminology of that note. In particular, it should be recalled that if T is an operator from a Hilbert space $\mathscr{H}$ to a Hilbert space $\mathscr{K}$, then T is said to be *affiliated with* a given ideal $\mathfrak{J}$ in $\mathscr{L}(\mathscr{H})$ if the operator $(T^*T)^{\frac{1}{2}}$ belongs to $\mathfrak{J}$. (In this paper, as in [1], all Hilbert spaces will be assumed to be *complex*, *separable*, and *infinite dimensional*, and all operators will be assumed to be *bounded* and *linear*. Furthermore, $\mathscr{L}(\mathscr{H})$ will denote the algebra of all operators on a Hilbert space $\mathscr{H}$, and all ideals in $\mathscr{L}(\mathscr{H})$ referred to will be *two-sided.*)

The following result is [1, Theorem 3. 1]. It is central to our present needs, and we restate it here for convenience of reference.

Theorem A. *Let $\mathfrak{J}$ be any ideal in $\mathscr{L}(\mathscr{H})$ other than the ideal of operators of finite rank, and let T be any operator on $\mathscr{H}$. Let λ be any fixed scalar in the boundary of the Calkin spectrum of T, and let ε be any positive number. Then there exists a decomposition of $\mathscr{H}$ into infinite dimensional subspaces $\mathscr{K}$ and $\mathscr{K}^{\perp}$ such that the restriction $(T-\lambda)|\mathscr{K}$ of $T-\lambda$ to $\mathscr{K}$ $((T-\lambda)|\mathscr{K}:\mathscr{K}\to\mathscr{H})$ is affiliated with the ideal $\mathfrak{J}$ and has norm less than ε.*

In [1], Theorem A was used to show that every operator in $\mathscr{L}(\mathscr{H})$ is unitarily equivalent to a particular kind of 2×2 operator matrix, and this result was then applied to obtain certain results in the theory of commutators. In this note, we again employ Theorem A, this time to show that every operator on a Hilbert space $\mathscr{H}$ is unitarily equivalent to a 3×3 operator matrix of a certain form (Theorem 2. 1); this result is then used to prove an interesting theorem concerning the ranges of derivations on $\mathscr{L}(\mathscr{H})$.

2. A matricial standard form. The purpose of this section is to prove the following rather surprising theorem.

Theorem 2. 1. *Let $\mathfrak{J}$ be any ideal in $\mathscr{L}(\mathscr{H})$ other than the ideal of operators of finite rank, and let T be any operator on $\mathscr{H}$. Let λ be any fixed scalar in the bound-*

ary of the Calkin spectrum of T, and let ε be any positive number. Then there exists a unitary isomorphism of $\mathscr{H}$ onto $\mathscr{H}\oplus\mathscr{H}\oplus\mathscr{H}$ which carries the operator $T-\lambda$ onto a 3×3 operator matrix (with entries from $\mathscr{L}(\mathscr{H})$) of the form

$$(*)\qquad \begin{pmatrix} J_{11} & A & B \\ J_{21} & C & D \\ J_{31} & J_{32} & J_{33} \end{pmatrix},$$

where $J_{11}, J_{21}, J_{31}, J_{32}$, and J_{33} all belong to the ideal $\mathfrak{J}$, and all have norm less than ε.

The proof of Theorem 2. 1 depends on Theorem A and the following elementary lemma.

Lemma 2. 2. *Let $\mathscr{M}$ and $\mathscr{N}$ be any two infinite dimensional subspaces of $\mathscr{H}$. Then there exist infinite dimensional subspaces $\mathscr{M}_1\subset\mathscr{M}$ and $\mathscr{N}_1\subset\mathscr{N}$ such that $\mathscr{M}_1$ and $\mathscr{N}_1$ are orthogonal.*

Proof. Let x_1 be any unit vector in $\mathscr{M}$. Then, since $\mathscr{N}$ has dimension greater than 1, there exists a unit vector y_1 in $\mathscr{N}$ that is orthogonal to x_1. Since $\mathscr{M}$ has dimension greater than 2, it follows that there exists a unit vector x_2 in $\mathscr{M}$ that is orthogonal to x_1 and to y_1. Continuing via an obvious induction argument, we obtain orthonormal sequences $\{x_n\}_{n=1}^{\infty}$ in $\mathscr{M}$ and $\{y_n\}_{n=1}^{\infty}$ in $\mathscr{N}$ such that for every pair j, k of positive integers, x_j is orthogonal to y_k. The proof is completed by taking for $\mathscr{M}_1$ and $\mathscr{N}_1$ the subspaces spanned by the sequences $\{x_n\}$ and $\{y_n\}$, respectively.

Proof of Theorem 2. 1. According to Theorem A, there exists an infinite dimensional subspace $\mathscr{M}$ of $\mathscr{H}$ such that $(T-\lambda)|\mathscr{M}$ is affiliated with the ideal $\mathfrak{J}$ and has norm less than ε. Furthermore, since $\bar{\lambda}$ belongs to the boundary of the Calkin spectrum of T^*, it also follows from Theorem A that there exists an infinite dimensional subspace $\mathscr{N}$ of $\mathscr{H}$ such that $(T^*-\bar{\lambda})|\mathscr{N}$ is affiliated with $\mathfrak{J}$ and has norm less than ε. If we now apply Lemma 2. 2 to $\mathscr{M}$ and $\mathscr{N}$, we obtain infinite dimensional subspaces $\mathscr{M}_1\subset\mathscr{M}$ and $\mathscr{N}_1\subset\mathscr{N}$ such that $\mathscr{M}_1$ and $\mathscr{N}_1$ are orthogonal. Furthermore, it is obvious that the choices of $\mathscr{M}_1$ and $\mathscr{N}_1$ can be made in such a way that $\mathscr{K}_1=(\mathscr{M}_1\oplus\mathscr{N}_1)^{\perp}$ is also infinite dimensional. Let $\tilde{\mathscr{H}}$ denote the threefold direct sum $\tilde{\mathscr{H}}=\mathscr{H}\oplus\mathscr{H}\oplus\mathscr{H}$, and let the subspaces $\mathscr{H}\oplus 0\oplus 0$, $0\oplus\mathscr{H}\oplus 0$, and $0\oplus 0\oplus\mathscr{H}$ of $\tilde{\mathscr{H}}$ be denoted by $\tilde{\mathscr{H}}_1$, $\tilde{\mathscr{H}}_2$, and $\tilde{\mathscr{H}}_3$, respectively. Choose φ to be any Hilbert space isomorphism of $\mathscr{H}$ onto $\tilde{\mathscr{H}}$ such that $\varphi(\mathscr{M}_1)=\tilde{\mathscr{H}}_1$, $\varphi(\mathscr{K}_1)=\tilde{\mathscr{H}}_2$, and $\varphi(\mathscr{N}_1)=\tilde{\mathscr{H}}_3$. Then it is clear that the operator $\tilde{T}=\varphi T\varphi^{-1}$ on $\tilde{\mathscr{H}}$ has the property that the restrictions $(\tilde{T}-\lambda)|\tilde{\mathscr{H}}_1$ and $(\tilde{T}-\lambda)^*|\tilde{\mathscr{H}}_3$ are both affiliated with the ideal $\mathfrak{J}$ and have norm less than ε. It follows easily (see, for example, Theorem 3. 1 of [1] and the remark following) that if $\tilde{T}-\lambda$ is written as a 3×3

matrix with entries from $\mathscr{L}(\mathscr{H})$ in the usual way, then all of the entries in the first column and third row of this matrix belong to the ideal $\mathfrak{J}$ and all have norm less than ε. Thus the proof is complete.

3. Application to derivations. In this section we apply Theorem 2. 1 to obtain a result concerning the ranges of derivations on $\mathscr{L}(\mathscr{H})$. Recall that such a derivation is a linear function D mapping $\mathscr{L}(\mathscr{H})$ into itself satisfying the equation $D(AB) = D(A)B+AD(B)$ for every pair A, B of operators on $\mathscr{H}$. It has been known for some time [4, Theorem 9] that every derivation on $\mathscr{L}(\mathscr{H})$ is an *inner derivation*; i.e., if D is such a derivation, then there exists an operator T on $\mathscr{H}$ such that $D(A) = TA-AT$ for every operator A in $\mathscr{L}(\mathscr{H})$. We shall indicate this relationship between a given derivation D and the operator T by writing $D=D_T$. (The operator T associated with D is not unique, since, if λ is any scalar, then $D_T=D_{T-\lambda}$.)

Theorem 3. 1. *Let $\mathfrak{J}$ be any ideal in $\mathscr{L}(\mathscr{H})$ other than the ideal $\mathfrak{F}$ of operators of finite rank. Then there exists no derivation D on $\mathscr{L}(\mathscr{H})$ whose range contains $\mathfrak{J}$.*

Proof. As noted above, we may assume that D is of the form $D=D_{T-\lambda}$, where T is some operator on $\mathscr{H}$ and λ is a fixed scalar in the boundary of the Calkin spectrum of T. It follows from [2, Theorem 4. 7] that there exists an ideal $\mathfrak{K}$ in $\mathscr{L}(\mathscr{H})$ such that $\mathfrak{F}\subsetneqq\mathfrak{K}\subsetneqq\mathfrak{J}$. Therefore, according to Theorem 2. 1, $T-\lambda$ is unitarily equivalent to a 3×3 operator matrix M acting on $\mathscr{H}\oplus\mathscr{H}\oplus\mathscr{H}$ with the property that all entries in the first column and all entries in the third row of M lie in $\mathfrak{K}$. Let J be an operator in $\mathfrak{J}$ that does not belong to $\mathfrak{K}$, and let J' be the operator on $\mathscr{H}$ whose image under the given unitary isomorphism between $\mathscr{H}$ and $\mathscr{H}\oplus\mathscr{H}\oplus\mathscr{H}$ is the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ J & 0 & 0 \end{pmatrix}.$$

Then clearly J' belongs to $\mathfrak{J}$, and, since the product (in either order) of M with every 3×3 matrix with entries from $\mathscr{L}(\mathscr{H})$ can be seen by calculation to have the property that its (3, 1) entry lies in the ideal $\mathfrak{K}$, it follows that the range of the derivation $D_{T-\lambda}$ does not contain J'; thus the proof is complete.

Note that the proof just concluded actually proves somewhat more than Theorem 3. 1. We include this stronger result as a proposition.

Proposition 3. 2. *Let $\mathfrak{J}$ be any ideal in $\mathscr{L}(\mathscr{H})$ other than the ideal of operators of finite rank, and let T be any operator on $\mathscr{H}$. Then for each fixed λ in the boundary of the Calkin spectrum of T, the linear manifold*

$$\{(T-\lambda)X-Y(T-\lambda): X,\quad Y\in\mathscr{L}(\mathscr{H})\}$$

fails to contain the ideal $\mathfrak{J}$.

4. Some comments. Although considerable progress has been made in commutator theory in the past few years, many questions concerning derivations remain unanswered. It is not known, for example, whether there exists a derivation on $\mathcal{L}(\mathcal{H})$ with the property that the identity operator lies in the (uniform) closure of its range. Furthermore, it is not known whether the ideal of finite rank operators is contained in the range of any derivation. (*Added in proof.* This point has also been settled in the negative by STAMPFLI.) Thus, it would appear that the topic of derivations on $\mathcal{L}(\mathcal{H})$ is an interesting area for continued investigation. In this connection it should be noted that J. G. STAMPFLI [5] has recently proved the pretty theorem that no derivation on $\mathcal{L}(\mathcal{H})$ has range that is norm dense in $\mathcal{L}(\mathcal{H})$. A different proof of this theorem can be given by using Theorem 2. 1 above. (It was known previously [3, Theorem 4] that derivation by the unilateral shift of multiplicity one has range dense in $\mathcal{L}(\mathcal{H})$ in the strong operator topology.)

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