# The spectral theorem for real Hilbert space

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## Introduction

The main purpose of this paper is to prove a spectral theorem for bounded normal operators in real Hilbert space. The self-adjoint case will follow as a corollary and is almost exactly the same as in the complex case. (However, the self-adjoint case for real Hilbert space is implicit in [3, pp. 269—276]). For normal operators, the theorem differs significantly from that for the complex case.

We begin by giving an example of a bounded normal operator in a real Hilbert space, which will turn out to be "essentially" the only example of a bounded normal operator in real Hilbert space. Consider $L_2(\mu)$ where $\mu$ is a measure with compact support defined on the Borel sets of the Euclidean plane. Further suppose that $\mu$ is symmetric about the $x$-axis, i.e., $\mu(e)=\mu(e^*)$ for each Borel set $e$, where $e^*$ is the reflection of $e$ about the $x$-axis. Then $L_2(\mu) = H_e \oplus H_0$, where $H_e$ consists of the $L_2(\mu)$ functions that are symmetric (even) about the $x$-axis, and $H_0$ consists of the $L_2(\mu)$ functions that are anti-symmetric (odd) about the $x$-axis. Consider $f$ in $L_2(\mu)$ as a function of $(r, \theta)$, and define the operator $T=T(\mu)$ on $L_2(\mu) = H_e \oplus H_0$ by

$$Tf = \begin{pmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} f_e \\ f_0 \end{pmatrix}.$$

## 1. The Spectral Theorem

Let $H$ be a real Hilbert space and let $A$ be an everywhere defined and bounded operator from $H$ into $H$, and in particular let $A$ be normal ($AA^*=A^*A$). Let $\bar{H}$ be the complexification of $H$, with elements $[x, y]$ ($x, y \in H$), and inner product $\langle [x, y], [t, z] \rangle = (x, t) - i(x, z) + i(y, t) + (y, z)$. Define $\bar{A}v = [Ax, Ay]$ if $v = [x, y]$. Then $\bar{A}$ is linear, bounded, and normal, with $\bar{A}^* v = [A^* x, A^* y]$ and $\|\bar{A}\| = \|A\|$.

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By the spectral theorem in complex Hilbert space, $\bar{A} = \int_{\sigma(A)} \lambda d\bar{E}$, where $\bar{E}$ is a self-adjoint measure, and $\sigma(\bar{A})$ is the spectrum of $\bar{A}$ in $\bar{H}$. Let $D$ be the disk in the plane with center 0 and radius $\|A\|$; then $D$ contains $\sigma(\bar{A})$.

We now compute $\bar{E}(e)$ for each Borel set $e$ contained in $D$. If $e$ is a compact set then $\chi_e$ (characteristic function of $e$) is the point limit of a bounded sequence of polynomials $p_n(z)$ in $z$ and $\bar{z}$, and hence $\bar{E}(e)v = \lim p_n(\bar{A})v$ for each vector $v$ in $\bar{H}$. But $(a+bi)(\bar{A})^n(\bar{A}^*)^m v = [aA^n A^{*m}x - bA^n A^{*m}y, aA^n A^{*m}y + bA^n A^{*m}x]$. Hence we deduce that $\bar{E}(e)$ is of the form $\bar{E}(e)v = [E_1(e)x - E_2(e)y, E_2(e)x + E_1(e)y]$ where $E_1(e)$, $E_2(e)$ are bounded operators.

Let $S$ be the collection of Borel subsets $e$ of $D$ such that $\bar{E}(e)(v) = [\Phi_1 x - \Phi_2 y, \Phi_2 x + \Phi_1 y]$ for all $v = [x, y] \in \bar{H}$, where $\Phi_1$ and $\Phi_2$ are bounded operators. From the above $S$ contains the compact sets, and it is easily verified that $S$ is a $\sigma$-ring. (To show $S$ is closed under complements one uses $\bar{E}(D) = I$, and to show $S$ is closed under intersections one uses $\bar{E}(e_1 \cap e_2) = \bar{E}(e_1)\cdot\bar{E}(e_2)$. To prove $S$ is closed under monotone limits use the fact that $\bar{E}(e)v = \lim \bar{E}(e_n)v$, where one considers $v$ of the form $[x, 0]$ and of the form $[0, y]$, and the uniform boundedness principle.)

So for each Borel set $e$ there exist unique bounded operators $E_1(e)$, $E_2(e)$ such that $\bar{E}(e)v = [E_1(e)y, E_2(e)x - E_2(e)x + E_1(e)y]$. We have $(\bar{E}(e))^* = \bar{E}(e)$, which is equivalent to saying that $(E_1(e))^* = E_1(e)$, and $(E_2(e))^* = -E_2(e)$, i.e. that $E_1(e)$ is self-adjoint and $E_2(e)$ is skew-symmetric. Since $E_2(e)$ is skew-symmetric, $(E_2(e)x, x) = 0$ for each $x$. Also, $\langle \bar{E}(e)v, v\rangle \geqq 0$ which implies $\mu(e) = (E_1(e)x, x) \geqq 0$ for all $x$. $(\bar{E}(e)v, w)$ is a regular Borel measure for each $v$, $w$ which implies that $\mu$ above is a regular non-negative Borel measure. Also, since $\left\|\left(\sum_{i=1}^{n} \lambda_i \bar{E}(e_i)\right)v\right\| \leqq \max |\lambda_i|\cdot\|v\|$ for any finite partition $\{e_1, e_2, \ldots, e_n\}$ of $D$, one has $\|\Sigma\lambda_i E_j(e_i)x\| \leqq \max |\lambda_i|\cdot\|x\|$ for $j = 1, 2$, so each $E_j$ is of bounded variation. This implies $\int f dE_j$ exists as a bounded operator from $H$ into $H$ for every bounded Borel measurable function $f$.

From the identity $\bar{E}(e_1 \cap e_2) = \bar{E}(e_1)\cdot\bar{E}(e_2)$ one obtains the identities

1) $E_1(e_1 \cap e_2) = E_1(e_1)E_1(e_2) - E_2(e_1)E_2(e_2)$,

2) $E_2(e_1 \cap e_2) = E_2(e_1)E_1(e_2) + E_1(e_1)E_2(e_2)$.

Since $\bar{E}(D) = I$, we have $E_1(D) = I$ and $E_2(D) = 0$. Also, $E_j\left(\bigcup_{i=1}^{\infty} e_i\right)x = \sum_i E_j(e_i)x$ for each $x$ in $H$ and for $j = 1, 2$ if the $\{e_i\}$ are disjoint. So $E_1$, $E_2$ are of bounded variation and countable additive in the strong operator topology.

From the spectral theorem we have $\bar{A}^n\bar{A}^{*m} = \int \lambda^{n-m} d\bar{E}$. Writing

$$\lambda = r(\cos\theta + i\sin\theta),$$

and expanding $\sum_{i=1}^{n} \lambda_i^n \bar{\lambda}_i^m \bar{E}(e_i)$ component-wise and taking a limit, we get $A^n A^{*m} = \int r^{n+m} \cos(n-m)\theta dE_1 - \int r^{n+m} \sin(n-m)\theta dE_2$, and $\int r^{n+m} \sin(n-m)\theta dE_1 = -\int r^{n+m} \cos(n-m)\theta dE_2$.

Lemma 1. *$E_1(e^*) = E_1(e)$ and $E_2(e^*) = -E_2(e)$.*

Proof. $0 = \int r^{n+m} \cos(n-m)\theta d(E_2 x, x) = -\int r^{n+m} \sin(n-m)\theta d\mu$, where $\mu(e) = (E_1(e)x, x)$. Let $f$ be continuous on $D$ and $f(r, -\theta) = -f(r, \theta)$. Given $\varepsilon > 0$, by the Stone—Weierstrass theorem there exists a trigonometric polynomial $p(r, \theta)$ such that $|f(r, \theta) - p(r, \theta)| < \varepsilon/2$ in $D$, where $p(r, \theta) = \Sigma a_{n,m} r^{n+m} \sin(n-m)\theta + \Sigma b_{n,m} r^{n+m} \cos(n-m)\theta$. Substituting $-\theta$ into the above inequality and adding show $|\Sigma b_{n,m} r^{n+m} \cos(n-m)\theta| < \varepsilon/2$. This implies $|f(r, \theta) - \Sigma a_{n,m} r^{n+m} \sin(n-m)\theta| < \varepsilon$. This gives $\int f(r, \theta) d\mu = 0$. If $e$ is a compact set lying entirely in the upper halfplane then there exists a bounded sequence $\{f_n\}$ of continuous functions converging pointwise to $\chi_e$ and vanishing off the upper half-plane. Define $g_n$ to equal $f_n$ in the upper half-plane and $-f_n(r, \theta)$ in the lower half-plane. Then $\int g_n d\mu = 0$, but $g_n$ converges pointwise to $\chi_e - \chi_{e^*}$, so by the dominated convergence theorem $\int (\chi_e - \chi_{e^*}) d\mu = 0$, or $\mu(e) = \mu(e^*)$. Let $S$ be the collection of Borel sets $e$ in $D$, lying in the upper half-plane and such that $\mu(e) = \mu(e^*)$. One can show that $S$ is a $\sigma$-ring containing the compact sets, so that $\mu(e) = \mu(e^*)$ for each Borel set $e$ lying in the upper half-plane. From this it easily follows that $\mu(e) = \mu(e^*)$ for every Borel set. $e$ Thus $(E_1(e)x, x) = (E_1(e^*)x, x)$ for each $x$, which implies $E_1(e) = E_1(e^*)$ since $E_1(e)$ is self-adjoint.

From the identity $\int r^{n+m} \cos(n-m)\theta dE_2 = -\int r^{n+m} \sin(n-m)\theta dE_1$, and since $E_1$ is symmetric about the $x$-axis and $r^{n+m} \sin(n-m)\theta$ is antisymmetric, we have $\int r^{n+m} \cos(n-m)\theta dE_2 = 0$; hence $\int r^{n+m} \cos(n-m)\theta d\nu = 0$, where $\nu(e) = (E_2(e)x, y)$. An argument similar to the above shows $\nu(e) = -\nu(e^*)$, i.e., $E_2(e) = -E_2(e^*)$.

Definition. Let $(E_1, E_2)$ be called a spectral pair provided 1) $E_1$ and $E_2$ are of finite variation and countably additive in the strong operator topology; 2) $E_1(e)$ is self-adjoint and $E_2(e)$ is anti-symmetric for each Borel set $e$; 3) $E_1(e) = E_1(e^*)$, and $E_2(e) = -E_2(e^*)$ for each Borel set $e$; 4) $E_1(e_1 \cap e_2) = E_1(e_1)E_1(e_1) - E_2(e_1)E_2(e_2)$, and $E_2(e_1 \cap e_2) = E_2(e_1)E_1(e_2) + E_1(e_1)E_2(e_2)$; 5) $E_1(D) = I$, $E_2(D) = 0$.

Summarizing the above:

Theorem 1. *If $A$ is a bounded normal operator on a real Hilbert space then there exists a unique spectral pair $(E_1, E_2)$ such that $Ax = \int r \cos\theta dE_1 x - \int r \sin\theta dE_2 x$ for all $x \in H$.*

Proof. The only remaining item to check is the uniqueness of the pair. Suppose $(E_1', E_2')$ is another such spectral pair. Let $\bar{E}'(e)v = [E_1'(e)x - E_2'(e)y, E_2'(e)x + E_1'(e)y]$. One can show directly that $\bar{E}'$ is a spectral measure and that $\bar{A} = \int \lambda d\bar{E}'$. By the spectral theorem in the complex case $\bar{E}' = \bar{E}$, and this implies $E_1' = E_1$, $E_2' = E_2$.

## 2. Spectral Representation

Theorem 1 will now be used to prove a spectral representation theorem, i.e., we will show that $H$ is the orthogonal direct sum of closed subspaces $\{H_\alpha\}$ where each $H_\alpha$ is isometrically isomorphic to an $L_2(\mu_\alpha)$ space and $A|H_\alpha$ is characterized on $L_2(\mu_\alpha)$ as the operator $T_\alpha$ described in the introduction.

Suppose there exists a cyclic vector $x \in H$ such that the linear span of the vectors of the form $A^n A^{*m} x$ is dense in $H$. Let $\mu(e) = (E_1(e)x, x)$. Then $\mu$ is a non-negative regular Borel measure, and $\mu(e) = \mu(e^*)$ for each Borel set $e$.

Recall that from the spectral theorem in the complex case we have $(\bar{A}^n)(\bar{A}^*)^m = \int \lambda^n \bar{\lambda}^m d\bar{E}$, so letting $\lambda = r(\cos + i \sin\theta)$ and expanding this component-wise give $A^n A^{*m} x = \int r^{n+m} \cos(n-m)\theta dE_1 x - \int r^{n+m} \sin(n-m)\theta dE_2 x$ and $(A^n x, A^m x) = \int r^{n+m} \cos(n-m)\theta d\mu = (f_n, f_m)$, where $f_n(r, \theta) = r^n(\cos n\theta + \sin n\theta)$. This follows since $f_n \cdot f_m = r^{n+m}(\cos(n-m)\theta + \sin(n+m)\theta)$, and $r^{n+m}\sin(n+m)\theta$ is an odd function in $\theta$. So one has $(A^n A^{*m} x, A^k A^{*g} x) = (f_{n+g}, f_{k+m}) = \int r^{n+m}(\cos(n-m)\theta + \sin(n-m)\theta) r^{k+g}(\cos(k-g)\theta + \sin(k-g)\theta) d\mu$.

If one defines $\Phi(\Sigma a_{nm} A^n A^{*m} x) = \Sigma a_{nm} r^{n+m}(\cos(n-m)\theta + \sin(n-m)\theta)$ then $\Phi$ is well-defined and is an isometry from the linear span of the $A^n A^{*m} x$ into $L_2(\mu)$. Moreover, its range is dense in $H$ since $\Phi(\tfrac{1}{2}(A^n A^{*m} + A^{*n} A^m)x) = r^{n+m}\cos(n-m)\theta$, $\Phi(\tfrac{1}{2}(A^n A^{*m} - A^{*n} A^m)x) = r^{n+m}\sin(n-m)\theta$, and the span of these functions is dense in $L_2(\mu)$. So $\Phi$ has a unique isometric extension of $H$ onto $L_2$.

Recall the operator $T$ defined in the Introduction. One can show by a straightforward calculation that $\Phi A \Phi^{-1} = T$ on the functions $r^{n+m}\cos(n-m)\theta$, $r^{n+m}\sin(n-m)\theta$, and hence for all of $L_2(\mu)$. Thus $A$ is "orthogonally equivalent" to $T$.

Theorem 2. *If $A$ is a bounded normal operator on the real Hilbert space $H$ and if $H$ contains a cyclic vector then there exists an $L_2(\mu)$ with $\mu(e) = \mu(e^*)$, such that $A$ is orthogonally equivalent to $T$ on $L_2(\mu)$* (see Introduction).

If there is no cyclic vector then apply Zorn's Lemma, see [1, pp. 910], to obtain $H = \oplus H_\alpha$ so that each $H_\alpha$ contains a cyclic vector $x_\alpha$.

Theorem 3. *Every bounded normal operator $A$ on the real Hilbert space $H$ is* *orthogonally equivalent to an orthogonal sum $\oplus T_\alpha$ of operators on spaces $L_2(\mu_\alpha)$ of the type defined in the introduction.*

Remark 1. If one defines

$$E_1(x) = \begin{pmatrix} f_1 & 0 \\ 0 & f_1 \end{pmatrix}, \quad E_2(e) = \begin{pmatrix} 0 & -f_2 \\ f_2 & 0 \end{pmatrix},$$

where $f_1$ is the even part of $\chi_e$, and $f_2$ is the odd part of $\chi_e$ then one can directly verify that $(E_1, E_2)$ is a spectral pair, and for continuous $f$, $T(f) = \left(\int r\cos\theta dE_1\right)(f) - \left(\int r\sin\theta dE_2\right)(f)$, and so from uniqueness, $(E_1, E_2)$ is the spectral pair for the operator $T$.

Remark 2. One could define a calculus for $A$ by defining $f(A) = \int f_1 dE_1 - \int f_2 dE_2$, where $f = f_1 + if_2$ with $f_1$ even, $f_2$ odd, and both Borel measurable. The details are similar to [1, pp. 895—902].

Remark 3. As a corollary to Theorem 1 one has the self-adjoint case (3, pp. 269—276). The unbounded case follows from the bounded case just as in (3, pp. 313—320). Also, one could now write out the unitary and skew-symmetric cases from Theorem 1 and Theorem 3. Also, one could easily show that for compact normal operators $H$ is the orthogonal direct sum of one and two dimensional invariant subspaces.

One could further use the above Theorem 1 in the skew-symmetric case and the methods found in (3, pp. 296—320) and (3, pp. 314—315) to obtain a spectral theorem for unbounded skew-symmetric operators in a real Hilbert space. Then using this theorem one could obtain Stone's theorem for real Hilbert space, see [2, pp. 38].

*Added in proofs.* The author has learned through private communication with Prof. TIN KIN WONG of Wayne State University that he has obtained some of the results of this paper by other methods. Also, Prof. Wong obtains the unbounded normal case by his methods. One could use the above methods and the unbounded self-adjoint and skew-symmetric cases to obtain the spectral theorem for unbounded normal operators.

## Bibliography

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