Slender modules, slender rings

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Introduction. The notion of slenderness is due to J. Łoś, who defined it originally for abelian groups in 1958 ([1], p. 169).

An abelian group is a Z-module, Z denoting the ring of integers, so that the theory of abelian groups appears as a special case of the theory of modules.

Conversely, a large part of the theory of abelian groups can be generalized to some classes of modules. The purpose of this paper is to indicate how the notion of slenderness can be extended in a natural way to modules and rings (considered as modules).

In a fundamental paper ([2], p. 71, Corollary 6), R. J. NUNKE has characterized all slender torsion free abelian groups: a torsion free abelian group is slender if and only if it contains no copy of the additive group Q of rational numbers, no copy of the additive group P of p-adic integers for any prime p, and no copy of the complete direct sum π of countably many infinite cyclic groups Z.

Similarly, the main problem in the theory of slender modules is to characterize all slender R-modules (for a given ring R), i.e. to investigate a theorem which may be the extension of Nunke's theorem to modules. It appears that such a generalization of Nunke's theorem to modules is not a trivial one and gives rise to hard problems of ring-theory and homological algebra. In this paper, we give only some *necessary* conditions for *any* module to be slender (Theorem 9), so that the problem of characterization of slender modules remains open. Finally, in § 4, we investigate some slender rings.

N.B. All rings considered in this paper are associative, commutative, with identity 1.

§ 1. Definition of slender modules and slender rings

1°) Let R be a ring. An R-module M is said to be *R-slender* (or, more briefly, *slender*) if for any non-zero homomorphism

$$h: \sum_{n=1}^{\infty}{}^{*} R_n \to M \qquad (R_n \cong R,\ n=1, 2, \dots)$$

we have $h(R_n)=0$ *for almost all n.*

2°) A ring R is said to be *slender* if R, considered itself as an R-module, is slender, i.e. if, for any non-zero homomorphism

$$h: \sum_{n=1}^{\infty}{}^{*} R_n \to R \qquad (R_n \cong R,\ n=1, 2, \ldots)$$

we have $h(R_n)=0$ *for almost all* n.

§ 2. Relation between slender modules over different rings

Theorem 1. *Let $R \subseteqq R'$ be two rings having the same identity 1. If an R'-module M is R-slender, then M is also R'-slender.*

Proof. Let the R'-module M be R-slender. Suppose that M is not R'-slender. Then there exists a non-zero homomorphism

$$h': \sum_{n=1}^{\infty}{}^{*} R'_n \to M \qquad (R'_n \cong R',\ n=1, 2, \ldots)$$

such that $h'(R'_n) \neq 0$ for infinitely many indices n. Then $h'(e_n)=h(e_n)\neq 0$ for infinitely many indices n, where

$$e_n = (\underbrace{0, \ldots, 0, 1}_{n}, 0, \ldots)$$

so that $h(R_n)\neq 0$ for infinitely many indices n, a contradiction to the hypothesis that M is R-slender. Hence M is R'-slender. Q.e.d.

Corollary 2. *A torsion free ring whose additive group is slender is slender itself.*

Proof. Let R' be a torsion free ring whose additive group is slender, $Z \subseteqq R'$. Since the Z-module R' is slender, from Theorem 1 we obtain that R'-module R' is slender. Q.e.d.

§ 3. Some necessary conditions for a module to be slender

Theorem 3. *Injective modules are not slender.*

Proof. Let M be an injective R-module. Consider the commutative diagram

$$\begin{tikzcd}
 & M & \\
0 \arrow[r] & \sum_{n=1}^{\infty} R_n \arrow[u, "f"] \arrow[r, "j"] & \sum_{n=1}^{\infty}{}^{*} R_n \arrow[ul, dashed, "h"']
\end{tikzcd}$$

where f is the non-zero homomorphism defined by

$$f\left(\sum_{i=1}^{K} r_i e_i\right) \overset{\mathrm{df}}{=} \left(\sum_{i=1}^{K} r_i\right)\cdot m$$

with a fixed non-zero element m of M, $e_i = (\underbrace{0, \dots, 0, 1}_{i}, 0, \dots) \in R_i$, and $r_i \in R$ $(i=1,2,\dots)$. In particular, we have $f(e_i) = hj(e_i) = h(e_i) = m \neq 0$ for every $i = 1, 2, \dots$. Hence M is not slender. Q.e.d.

Theorem 4. *Quotient modules of injective modules are not slender.*

Proof. Let M be an injective R-module, $N(\neq M)$ a submodule and $m \in M \setminus N$. Consider the commutative diagram

$$\begin{tikzcd}
 & M \arrow[r, "p"] & M/N \\
0 \arrow[r] & \sum_{n=1}^{\infty} R_n \arrow[u, "f"] \arrow[r, "j"] & \sum_{n=1}^{\infty}{}^{*} R_n \arrow[ul, dashed, "h"'] \arrow[u, dashed, "g"']
\end{tikzcd}$$

where f is the non-zero homomorphism defined in the proof of Theorem 3. We have $g(e_i) = ph(e_i) = p(m) \neq 0$ for every $i = 1, 2, \dots$. Hence M/N is not slender. Q.e.d.

Corollary 5. *Vector spaces are not slender.*

Proof. It is known that every module over a field (i.e. every vector space) is injective.

Corollary 6. *Fields are not slender.*

Theorem 7. *The ring P of p-adic integers is not slender.*

Proof. The homomorphism

$$\sum_{n=1}^{\infty}{}^{*} P_n \to P \qquad (P_n \cong P;\ n = 1, 2, \dots)$$

defined by

$$(\pi_1, \pi_2, \dots) \to \sum_i \pi_i p^i \in P$$

is such that

$$e_i = (\underbrace{0, \dots, 0, 1}_{i}, 0, \dots) \to p^i \neq 0$$

for every $i = 1, 2, \dots$. Hence P is not slender. Q.e.d.

Theorem 8. *Let R be a ring. Then the complete direct sum* $\sum_{n=1}^{\infty *} R$ *is not R-slender.*

Proof. The proof results from the simple fact that the identity transformation

$$\mathrm{id}: \sum_{n=1}^{\infty *} R \to \sum_{n=1}^{\infty *} R$$

is a homomorphism.

Now, we summarize the results of this paragraph. Since a submodule of a slender module is again slender, Theorem 3 and Theorem 8 give us the following *necessary* conditions for *any* module to be slender:

Theorem 9. *Let R be a ring. If an R-module M is R-slender, then it contains no copy of any injective R-module and no copy of the complete direct sum* $\sum_{n=1}^{\infty *} R$.

§ 4. Some slender rings

Consider now the following problem:

Find all slender torsion free rings of rank 1.

It is known that every torsion free abelian group of rank 1, not isomorphic to the additive group Q of rational numbers, is slender. If we combine this result and Corollary 2, we obtain the solution of the above problem:

Theorem 10. *Every torsion free ring of rank 1, not isomorphic to the ring Q of rational numbers, is slender.*

The structure of all torsion free rings of rank 1 is completely known (cf. [3]), so that we obtain, in this way, an important class of slender rings. In particular, the ring $Z_{(p)} = \left\{\frac{k}{p^n}\right\}$ (p prime; $k \in Z$) and the ring Z of integers are slender.

References

[1] L. Fuchs, *Abelian Groups* (Budapest, 1958).
[2] R. J. Nunke, Slender Groups, *Acta Sci. Math.*, **23** (1962), 67—73.
[3] L. Rédei and T. Szele, Die Ringe „ersten Ranges", *Acta Sci. Math.*, **12 A** (1950), 18—29.

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(Received April 9, 1970)