Basic study of general products and homogeneous homomorphisms. I

By TAKAYUKI TAMURA in Davis (California, U.S.A.)*)

§ 1. Intruduction

We frequently meet the problem to study semigroups S which are homomorphic onto a given semigroup T. Of course the problem in this form is too vague to be treated in general. Let us restrict ourselves to the following problem:

Given a semigroup T, study semigroups S such that S is homomorphic onto T under a map f and such that the cardinal number of the inverse image set of each element of T is constant, i.e. given m

$$|tf^{-1}|=m \quad \text{for all} \quad t\in T.$$

Such a homomorphism of S is called a *homogeneous homomorphism*. Let A be a set with cardinality m. We will introduce a concept "general product" of a set A by a semigroup T, which will be equivalent to the concept of homogeneous homomorphism. This concept includes the various known concepts. Then the first problem proposed above will be connected with the second restricted problem, that is, if S is homomorphic to T then the homomorphism can be extended to a homogeneous homomorphism of certain semigroup S' to T. Related to general product, we will consider the system of all binary operations defined on a set.

A part of the outline of this paper was reported in [11], [12] without proof. This paper is to report basic results of general products but its development and applications will be reported as the continuation in the future. Computational results related to this paper will be separately reported though a part of those were done in [12].

*) This research was partly supported by NSF GP-5988 and GP-7608.

§ 2. The system of operations

Let E be a set and $\mathscr{B}_E$ be the set of all binary operations defined on E. Let $x, y \in E$, $\theta \in \mathscr{B}_E$ and let $x\theta y$ denote the product of x and y by θ. A groupoid with θ defined on E is denoted by $E(\theta)$. The equality of elements $\theta, \eta \in \mathscr{B}_E$ is defined as follows:

$$\theta = \eta \quad \text{if and only if} \quad x\theta y = x\eta y \quad \text{for all} \quad x, y \in E.$$

Let $a \in E$ be fixed. Two binary operati ons ${}_a\Pi$ and Π_a are defined in $\mathscr{B}_E$ as follows:

(1) $\quad x(\theta_a\Pi\eta)y = (x\theta a)\eta y \quad \text{and} \quad x(\theta\Pi_a\eta)y = x\theta(a\eta y) \quad \text{for} \quad \theta, \eta \in \mathscr{B}_E,\ x, y \in E.$

It is clear that θ is associative if and only if $\theta a^*\theta = \theta^* a\theta$ for all $a \in E$.

Proposition 1. *$\mathscr{B}_E$ is a semigroup with respect to ${}_a\Pi$ and Π_a for all $a \in E$.*

Proof. *To prove* $(\xi_a\Pi\eta)a^*\theta = \xi_a\Pi(\eta_a\Pi\theta)$, $\xi, \eta, \theta \in \mathscr{B}_E$. We have

$$x[(\xi_a\Pi\eta)_a\Pi\theta]y = \{x(\xi_a\Pi\eta)a\}\theta y = \{(x\xi a)\eta a\}\theta y =$$

$$= (x\xi a)(\eta_a\Pi\theta)y = x[\xi_a\Pi(\eta_a\Pi\theta)]y \quad \text{for all} \quad x, y \in E.$$

Likewise we can prove $(\xi\Pi_a\eta)\Pi_a\theta = \xi\Pi_a(\eta^* a\theta)$.

The semigroups $\mathscr{B}_E$ with ${}_a\Pi$ and Π_a are denoted by $\mathscr{B}_E({}_a\Pi)$ or $\mathscr{B}({}_a\Pi)$, and $\mathscr{B}_E(\Pi_a)$ or $\mathscr{B}(\Pi_a)$ respectively.

Let φ be a permutation of E. For $\theta \in \mathscr{B}_E$, $\theta\varphi$ is defined as follows:

(2) $$x(\theta\varphi)y = [(x\varphi^{-1})\theta(y\varphi^{-1})]\varphi$$

or, by substituting x for $x\varphi^{-1}$,

(3) $$(x\theta y)\varphi = (x\varphi)(\theta\varphi)(y\varphi).$$

The mapping $\theta \to \theta\varphi$ is a permutation of $\mathscr{B}_E$. For any $\eta \in \mathscr{B}_E$, define θ by

$$x\theta y = [(x\varphi)\eta(y\varphi)]\varphi^{-1}.$$

Then we can easily prove $\theta\varphi = \eta$. Hence the mapping $\theta \to \theta\varphi$ is onto. To prove one-to-one. Suppose $\theta\varphi = \eta\varphi$. Then

$$[(x\varphi^{-1})\theta(y\varphi^{-1})]\varphi = [(x\varphi^{-1})\eta(y\varphi^{-1})]\varphi.$$

Since φ is a permutation of E, we have $(x\varphi^{-1})\theta(y\varphi^{-1}) = (x\varphi^{-1})\eta(y\varphi^{-1})$ where $x\varphi^{-1}$, $y\varphi^{-1}$ run throughout E and hence $\theta = \eta$.

Thus φ induces a permutation of $\mathscr{B}_E$. This permutation is still denoted by φ.

Proposition 2. *$(\theta_a\Pi\eta)\varphi = (\theta\varphi)_{a\varphi}\Pi(\eta\varphi)$ and $(\theta\Pi_a\eta)\varphi = (\theta\varphi)\Pi_{a\varphi}(\eta\varphi)$.*

Proof. we have

$$x(\theta_a\Pi\eta)\varphi y=[x\varphi^{-1}(\theta_a\Pi\eta)y\varphi^{-1}]\varphi=[\{(x\varphi^{-1})\theta a\}\eta(y\varphi^{-1})]\varphi=$$

$$=[x(\theta\varphi)(a\varphi)](\eta\varphi)y=x[(\theta\varphi)_{a\varphi}\Pi(\eta\varphi)]y \quad \text{for} \quad x, y\in E.$$

For $\theta\in\mathscr{B}_E$ we define θ' as follows:

(4) $$x\theta' y=y\theta x.$$

Then we have

Proposition 3. $(\theta_a\Pi\eta)'=\eta'\Pi_a\theta'$.

Proof. For all $x, y\in E$, $x(\theta_a\Pi\eta)'y=y(\theta_a\Pi\eta)x=(y\theta a)\eta x=xy'(y\theta a)=x\eta'(a\theta' y)=$ $=x(\eta'\Pi_a\theta')y$.

Proposition 4. *$\mathscr{B}({}_a\Pi)\cong\mathscr{B}({}_b\Pi)$ and $\mathscr{B}(\Pi_a)\cong\mathscr{B}(\Pi_b)$ for all $a, b\in E$. Furthermore, $\mathscr{B}({}_a\Pi)$ is anti-isomorphic with $\mathscr{B}(\Pi_a)$.*

Proof. Let φ be a permutation of E such that $a\varphi=b$. By Proposition 2, $(\theta_a\Pi\eta)\varphi=(\theta\varphi)_b\Pi(\eta\varphi)$. This shows that φ is an isomorphism of $\mathscr{B}({}_a\Pi)$ onto $\mathscr{B}({}_b\Pi)$. Similarly we have by the second part of Proposition 2 that φ is an isomorphism of $\mathscr{B}(\Pi_a)$ onto $\mathscr{B}(\Pi_b)$.

§ 3. General product of a set by a groupoid

Let S be a set and T be a groupoid. Consider a mapping Θ of $T\times T$ into $\mathscr{B}_S$:

$$(\alpha, \beta)\Theta=\theta_{\alpha,\beta}, \qquad (\alpha, \beta)\in T\times T.$$

Now $S\times T = \{(x, \alpha);\ x\in S,\ \alpha\in T\}$ in which $(x, \alpha)=(y, \beta)$ if and only if $x=y$, $\alpha=\beta$. Given S, T, Θ, a binary operation is defined on $S\times T$ as follows:

(5) $$(x, \alpha)(y, \beta)=(x\theta_{\alpha,\beta}y, \alpha\beta).$$

Definition. The groupoid $S\times T$ with (5) is called a *general product* of a set S by a groupoid T with respect to Θ, and is denoted by $S\overline{\times}_\Theta T$. If it is not necessary to specify Θ, it is denoted by $S\overline{\times}T$.

Definition. If a groupoid D is isomorphic onto some $S\overline{\times}_\Theta T$, $|S|>1$, $|T|>1$, then D is called *general-product decomposable (gp-decomposable)*.

Immediately we see that $S\overline{\times}_\Theta T$ is homomorphic onto T by the mapping $p:(x, \alpha)\to\alpha$. This mapping is called the projection of $S\overline{\times}T$ onto T. Likewise we can define the projection of $S\overline{\times}T$ onto S.

Proposition 5. *$S\bar{\times}_\Theta T$ is a semigroup if and only if T is a semigroup and*

$$\theta_{\alpha,\beta\,a}\Pi\theta_{\alpha\beta,\gamma}=\theta_{\alpha,\beta\gamma}\Pi_a\theta_{\beta,\gamma} \quad \text{for all} \quad a\in S, \quad \text{all} \quad \alpha,\beta,\gamma\in T.$$

Proof. The proposition is immediately proved as follows:

$$[(x,\alpha)(y,\beta)](z,\gamma)=(x\theta_{\alpha,\beta}y,\alpha\beta)(z,\gamma)=((x\theta_{\alpha,\beta}y)\theta_{\alpha\beta,\gamma}z,(\alpha\beta)\gamma),$$
$$(x,\alpha)[(y,\beta)(z,y)]=(x,\alpha)(y\theta_{\beta,\gamma}z,\beta\gamma)=(x\theta_{\alpha,\beta\gamma}(y\theta_{\beta,\gamma}z),\alpha(\beta\gamma)).$$

Definition. Let g be a homomorphism of a groupoid D onto a groupoid T: $D=\bigcup_{\alpha\in T}D_\alpha$; $D_\alpha g=\alpha$. If either $|D_\alpha|=1$ for all α or if $|T|=1$, g is called *trivial*; otherwise g is called *proper*. If $|D_\alpha|=|D_\beta|$ for all $\alpha,\beta\in T$, then g is called a *homogeneous homomorphism* (h-homomorphism) of D, or D is said to be *homogeneously homomorphic* (h-homomorphic) onto T.

Theorem 6. *A groupoid D is isomorphic onto $S\bar{\times}_\Theta T$ for some S and some Θ if and only if D is h-homomorphic onto T. More precisely, D is gp-decomposable if and only if D has a proper h-homomorphism.*

Proof. Suppose that D is homogeneously homomorphic onto T under g:

$$D=\bigcup_{\alpha\in T}D_\alpha,\quad D_\alpha g=\alpha.$$

Let S be a set with $|S|=|D_\alpha|$ for all $\alpha\in T$, and f_α be a one-to-one mapping of D_α onto S. After fixing a system $\{f_\alpha;\ \alpha\in T\}$, for each $(\alpha,\beta)\in T\times T$ we define a binary operation $\theta_{\alpha,\beta}$ on S as follows: Let

$$x\theta_{\alpha,\beta}y=[(xf_\alpha^{-1})(yf_\beta^{-1})]f_{\alpha\beta} \tag{6}$$

$x,y\in S$, where $\alpha\beta$ is the product in T. Now

$$D=\bigcup_{\alpha\in T}D_\alpha \quad \text{where} \quad D_\alpha=\{x\in D;\, xg=\alpha\}.$$

Let a be any element of D, hence $a\in D_\alpha$ for some $\alpha\in T$. We define a mapping ψ of D onto $S\times T$ as follows: $a\psi=(af_\alpha,\alpha)$. Then ψ is one-to-one: suppose $(af_\alpha,\alpha)=(bf_\beta,\beta)$. By the definition of equality we have $\alpha=\beta$, $af_\alpha=bf_\beta$. Since f_α is one-to-one, $a=b$. It is clear that ψ is onto. We shall prove $(ab)\psi=(a\psi)(b\psi)$. Let $a\in D_\alpha$, $b\in D_\beta$. By (6)

$$(ab)\psi=((ab)f_{\alpha\beta},\alpha\beta)=((af_\alpha)\theta_{\alpha,\beta}(bf_\beta),\alpha\beta)=(af_\alpha,\alpha)(bf_\beta,\beta)=(a\psi)(b\psi).$$

Consequently $D\cong S\bar{\times}_\Theta T$.

Conversely, suppose D is isomorphic with $S\bar{\times}_\Theta T$ under a mapping $f\colon D\to S\bar{\times}_\Theta T$. Let p be the projection of $S\bar{\times}_\Theta T$ onto T:

$$(x,\alpha)\xrightarrow{p}\alpha.$$

Then fp is a homomorphism of D onto T. Let $D_\alpha = \{a \in D;\ a(fp) = \alpha\}$, $D'_\alpha = \{(x, \alpha);\ x \in S\}$. Since f is one-to-one, $|D_\alpha| = |D'_\alpha| = |S|$ for all $\alpha \in T$. This completes the proof of the theorem.

As seen in the proof of Theorem 6, even if D, S, T are given, Θ depends on the choice of $\{f_\alpha; \alpha \in T\}$. Suppose that for given D, S, T,

$$\Theta : \{\theta_{\alpha,\beta};\ (\alpha, \beta) \in T \times T\} \text{ is determined by } \{f_\alpha;\ \alpha \in T\},$$

$$\Theta' : \{\theta'_{\alpha,\beta};\ (\alpha, \beta) \in T \times T\} \text{ is determined by } \{f'_\alpha;\ \alpha \in T\}.$$

What relationship is there between Θ and Θ'?

To state the problem generally we need to introduce some terminology:

Definition. Let g and g' be homomorphisms of groupoids A and B onto a groupoid C. An isomorphism h of A into (onto) B is called a restricted isomorphism of A into (onto) B with respect to g and g' or A is restrictedly isomorphic into (onto) B with respect to g and g' if there is a permutation k of C such that the following diagram is commutative:

$$h \cdot g' = g \cdot k \qquad \begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ B & \xrightarrow[g']{} & C \end{array}$$

The permutation k of C is an automorphism of C. Let $\alpha, \beta \in C$. $\alpha = xg$, $\beta = yg$ for some $x, y \in A$. We have

$$(\alpha\beta)k = [(xg)(yg)]k = [(xy)g]k = [(xy)h]g' = (xhg')(yhg') = [(xg)k][(yg)k] = (\alpha k)(\beta k).$$

Now the problem is this: Given S and T, let $D = S \overline{\times}_\Theta T$, $D' = S \overline{\times}_{\Theta'} T$. Let p and p' be the projections of D and D' onto T respectively. Under what condition on Θ and Θ' is D restrictedly isomorphic onto D' with respect to p and p'?

Definition. Let $G(\theta)$ and $G'(\theta')$ be groupoids with binary operations θ, θ' respectively. If there are three one-to-one mappings h, q, r of $G(\theta)$ onto $G'(\theta')$ such that $(x\theta y)r = (xh)\theta'(yq)$ for all $x, y \in G(\theta)$, then we say that $G(\theta)$ is *isotopic* to $G'(\theta')$ (see [1]). If it is necessary to specify h, q, r, we say $G(\theta)$ is (h, q, r)-isotopic to $G'(\theta')$. We denote it by

$$G(\theta) \underset{(h,q,r)}{\approx} G'(\theta') \quad \text{or} \quad G(\theta) \approx G'(\theta').$$

Theorem 7. *Let S and T be fixed. Let $(\alpha, \beta)\Theta = \theta_{\alpha,\beta}$, $(\alpha, \beta)\Theta' = \theta_{\alpha,\beta}$, $\alpha, \beta \in T$. $S \overline{\times}_\Theta T$ is restrictedly isomorphic with $S \overline{\times}_{\Theta'} T$ with respect to p, p' if and only if there is an automorphism $\alpha \to \alpha'$ of T and there is a system $\{f_\alpha;\ \alpha \in T\}$ of permutations of S such that $S(\theta_{\alpha,\beta})$ is $(f_\alpha, f_\beta, f_{\alpha\beta})$-isotopic to $S(\theta'_{\alpha',\beta'})$ for all $\alpha, \beta \in T$.*

Proof. Let h be a restricted isomorphism $S\times_{\theta}T\to S\times_{\theta'}T$ with respect to the projections p, p' as follows:

$$\begin{array}{ccc} S\overline{\times}_{\theta}T & \xrightarrow{p} & T \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ S\overline{\times}_{\theta'}T & \xrightarrow[p']{} & T \end{array}$$

Let $(x, \alpha)\in S\overline{\times}_{\theta}T$ and $(x, \alpha)h=(x', \alpha')\in S\overline{\times}_{\theta'}T$. Immediately $\alpha'=\alpha k$, further $(x, \alpha)h=(y, \beta)h$ implies $x=y$ and $\alpha=\beta$. Thus $x\to x'$ is a permutation of S depending on α. This permutation is denoted by l_{α} and then $(x, \alpha)h=(xl_{\alpha}, \alpha')$, where $\alpha'=\alpha k$. By using this notation,

$$[(x, \alpha)(y, \beta)]h=(x\theta_{\alpha,\beta}y, \alpha\beta)h=((x\theta_{\alpha,\beta}y)l_{\alpha\beta}, (\alpha\beta)'),$$

$$(x, \alpha)h\cdot(y, \beta)h = (xl_{\alpha}, \alpha k)(yl_{\beta}, \beta k) = ((xl_{\alpha})\theta'_{\alpha',\beta'}(yl_{\beta}), \alpha'\beta')$$

and we have $(\alpha\beta)'=\alpha'\beta'$, $(x\theta_{\alpha,\beta}y)l_{\alpha\beta}=(xl_{\alpha})\theta'_{\alpha',\beta'}(yl_{\beta})$. Therefore

$$S(\theta_{\alpha,\beta}) \underset{(l_{\alpha}, l_{\beta}, l_{\alpha\beta})}{\approx} S(\theta'_{\alpha',\beta'}) \quad \text{for all} \quad \alpha, \beta\in T. \tag{7}$$

Conversely suppose there is an automorphism $k:\alpha\to\alpha'$ of T and a system $\{l_{\alpha};\ \alpha\in T\}$ of permutations of S satisfying (7). We define a mapping h of $S\overline{\times}_{\theta}T$ onto $S\overline{\times}_{\theta'}T$ as follows: $(x, \alpha)h=(xl_{\alpha}, \alpha')$. Then we can easily see that h is one-to-one and $[(x, \alpha)(y, \beta)]h = (x, \alpha)h\cdot(y, \beta)h$. To prove that h is a restricted isomorphism, observe that $(x, \alpha)hp'=(xl_{\alpha}, \alpha')p'=\alpha'$ and $(x, \alpha)pk=\alpha k=\alpha'$ for all $(x, \alpha)\in S\overline{\times}_{\theta}T$; hence $hp'=pk$. Thus the proof of the theorem is completed.

As usual the product $\varrho\cdot\sigma$ of binary relations ϱ, σ on D is defined by

$$\varrho\cdot\sigma = \{(x, y);\ (x, z)\in\varrho, (z, y)\in\sigma \quad \text{for some} \quad z\in D\}.$$

Let $\omega = D\times D$, $\iota=\{(x, x);\ x\in D\}$.

The following theorem characterizes general product in terms of relations.

Theorem 8. *A groupoid D is gp-decomposable if and only if there is a congruence ϱ on D and an equivalence σ on D such that $\varrho\neq\omega$, $\sigma\neq\omega$,*

$$\varrho\cdot\sigma = \omega, \quad \textit{and} \tag{8}$$

$$\varrho\cap\sigma=\iota, \tag{9}$$

in which (8) *can be replaced by $\sigma\cdot\varrho = \omega$. Then $D \cong (D/\sigma)\overline{\times}(D/\varrho)$.*

Proof. Suppose $D \cong S\overline{\times}_{\theta}T$. Let ϱ be the congruence induced by the homomorphism $g:D\to T$. As stated in the proof of Theorem 1, $D=\bigcup_{\alpha\in T} D_{\alpha}$ where $|D_{\alpha}|=|S|$.

Let f_α be a one-to-one mapping of D_α onto S. Now we define a relation σ on D as follows: $x\sigma y$ if and only if $xf_\alpha = yf_\beta$ for α and β such that $x \in D_\alpha$, $y \in D_\beta$.

Now take $a, b \in D$ arbitrarily and assume $a \in D_\alpha$, $b \in D_\beta$. Let $c = bf_\beta f_\alpha^{-1}$. Then $c \in D_\alpha$, and $a\varrho c$, $c\sigma b$. Thus we have proved $\varrho \cdot \sigma = \omega$. Suppose $a\varrho b$ and $a\sigma b$, that is, $a, b \in D_\alpha$ and $af_\alpha = bf_\alpha$. Since f_α is one-to-one, $a = b$; therefore $\varrho \cap \sigma = \iota$.

Conversely, suppose that there is a congruence ϱ, $\varrho \neq \omega$, and an equivalence σ, $\sigma \neq \omega$, on D such that (8) and (9) are satisfied. Let T be the factor groupoid D/ϱ and S be the factor set D/σ. Let A and B be any ϱ-class and σ-class respectively and let $x \in A$, $y \in B$. By (8) there is $z \in D$ such that $x\varrho z$ and $z\sigma y$. This means that $A \cap B \neq \emptyset$. Suppose $x\varrho z$, $z\sigma y$, $x\varrho z'$ and $z'\sigma y$. Then $z\varrho z'$ and $z\sigma z'$. By (9), we have $z = z'$. Thus $A \cap B$ consists of exactly one element. Therefore the cardinal number of each ϱ-class is equal. By Theorem 6, we have $D \cong S \overline{\times} T$.

§ 4. Examples

The following well-known concepts are regarded as examples of general product.

Example 1. *Direct Product.* Suppose Θ maps (α, β) to a constant element θ, that is, $(\alpha, \beta)\Theta = \theta$ for all $\alpha, \beta \in T$. Then θ is automatically associative by Proposition 5. In other words S is a semigroup with θ. Thus $S \overline{\times}_\Theta T$ is the direct product of S and T.

Example 2. *Semi-direct product* (see [3], [6], [7]). Let S and T be semigroups, and Y be a homomorphism of T into the endomorphism semigroup of S, $t \xrightarrow{Y} Y_t$. The semi-direct product of S by T with respect to Y is the set $S \times T$ with the operation

$$(s_1, t_1)(s_2, t_2) = (s_1(Y_{t_1}(s_2)), t_1 t_2).$$

This is regarded as $S \overline{\times}_\Theta T$ in which $s_1 \theta_{t_1, t_2} s_2 = s_1 \cdot Y_{t_1}(s_2)$.

Example 3. *Rees' regular representation of completely simple semigroups* (see [2]). Let G be a group and F be a rectangular band

$$F = \{(\lambda, \mu);\ \lambda \in \Lambda, \mu \in M\},\ (\lambda, \mu)(\nu, \xi) = (\lambda, \xi).$$

Let $P = (p_{\mu\lambda})$, $\mu \in M$, $\lambda \in \Lambda$ be a matrix over G. If we define Θ by

$$x\theta_{(\lambda,\mu)(\nu,\xi)} y = x p_{\mu\nu} y,$$

then $G \overline{\times}_\Theta F$ is a completely simple semigroup.

Example 4. *Commutative archimedean cancellative semigroups without idempotent* (see [9]). Such a semigroup is called an $\mathfrak{N}$-semigroup. Let G be an abelian group and N be the set of all non-negative integers. Suppose a function $I: K\times K \to N$ satisfies

(a) $I(\alpha, \beta)=I(\beta, \alpha)$.

(b) $I(\alpha, \beta)+I(\alpha\beta, \gamma) = I(\alpha, \beta\gamma)+I(\beta, \gamma)$.

(c) $I(\varepsilon, \varepsilon)=1$, ε being the identity element of G.

(d) For every $\alpha\in G$ there is a positive integer m such that $I(\alpha^m, \alpha)>0$.

Define an operation on the set $S = N\times G$ by

$$(m, \alpha)(n, \beta) = (m+n+I(\alpha, \beta), \alpha\beta)$$

Then S is an $\mathfrak{N}$-semigroup. Every $\mathfrak{N}$-semigroup is obtained in this manner.

$$S \cong N \overline{\times}_{\Theta} G \text{ where } m\theta_{\alpha,\beta}n = m+n+I(\alpha, \beta).$$

Example 5. *Group extensions* (see [3], [5]). Let N and H be groups and let G be the group extension of N by H. A mapping $\alpha\to f_\alpha$ associates with each $\alpha\in H$ an automorphism f_α of N such that $f_{\alpha\beta}(x)=f_\alpha f_\beta(x)$, $x\in N$. Consider another mapping, $(\alpha, \beta)\to c_{\alpha,\beta}$ of $H\times H$ into N such that

$$c_{\alpha,\beta}f_{\alpha\beta}(x)c_{\alpha\beta,\gamma}=f_{\alpha\beta}(x)f_\alpha(c_{\beta,\gamma})c_{\alpha,\beta\gamma}$$

for all $\alpha, \beta, \gamma\in H$ and all $x\in N$. Then

$$G\cong N\overline{\times}_{\Theta}H \quad \text{where} \quad x\theta_{\alpha,\beta}y=xf_\alpha(y)c_{\alpha,\beta}.$$

Example 6. *Schreier Extension.* Let A and B be commutative semigroups with identity element. A Schreier extension of A by B in HANCOCK's sense [4] or RÉDEI's sense is an example of general product. Examples 4 and 5 are Schreier extensions.

Example 7. *H-semigroups* (see [10]).

As an extremely special case finite semigroups S having property that all homomorphisms of S are homogeneous were studied.

§ 5. Left (right) general product

Definition. A general product $S\overline{\times}_{\Theta}T$ is called a *left general product* of S by T if

(10) $$(\alpha, \beta)\Theta=(\alpha, \gamma)\Theta \quad \text{for all} \quad \alpha, \beta, \gamma\in T.$$

$S\overline{\times}_{\Theta}T$ is called a *right general product* of S by T if

(11) $$(\alpha, \beta)\Theta=(\gamma, \beta)\Theta \quad \text{for all} \quad \alpha, \beta, \gamma\in T.$$

In the case (10) $\theta_{\alpha,\beta}$ depends on only α, so $\theta_{\alpha,\beta}$ is denoted by θ_{α}. Then (5) becomes

(12) $$\theta_{\alpha\cdot a}\Pi\theta_{\alpha\beta\cdot}=\theta_{\alpha\cdot}\Pi_a\theta_{\beta\cdot} \text{ for all } \alpha,\beta\in T, \text{ all } a\in S.$$

In the case (11), $\theta_{\alpha,\beta}$ is denoted by $\theta_{\cdot\beta}$ and we have

(13) $$\theta_{\cdot\alpha a}\Pi\theta_{\cdot\beta}=\theta_{\cdot\alpha\beta}\Pi_a\theta_{\cdot\beta} \text{ for all } \alpha,\beta\in T, \text{ all } a\in S.$$

σ By a left congruence we mean a left compatible equivalence, i.e. an equivalence satisfying

(14) $$x\sigma y\Rightarrow zx\ \sigma\ zy \quad \text{for all} \quad z.$$

Theorem 9. *Let D be a groupoid. D is isomorphic with a left general product of a set S, $|S|>1$, by a groupoid T, $|T|>1$, if and only if there is a congruence ϱ on D and a left congruence σ on D such that $\varrho\neq\omega$, $\sigma\neq\omega$, $D/\varrho\cong T$, $|D/\sigma|=|S|$, and $\varrho\cdot\sigma=\omega$, $\varrho\cap\sigma=\iota$.*

Proof. Theorem 8 is applicable to this theorem except for (14). Suppose D is isomorphic with a left general product of S by T under a map h. Let $g=hp$ and $f=hq$

$$D\xrightarrow{h}S\overline{\times}T\xrightarrow{p}T$$
$$\searrow^{q}$$
$$S$$

Let f_α denote the restriction of f to D_α and let $D_\alpha g=\alpha$, $(x,\alpha)\in S\times T$, $(x,\alpha)p=\alpha$, $(x,\alpha)q=x$; ϱ and σ are defined as in the first part of the proof of Theorem 8. We need prove (14) only. Suppose $a, b, c\in D$ and $b\sigma c$ and let

$$ah=(x,\alpha),\quad bh=(y,\beta),\quad ch=(y,\gamma).$$

Then $(x,\alpha)(y,\beta)=(x\theta_{\alpha\cdot}y,\alpha\beta)$, $(x,\alpha)(y,\gamma)=(x\theta_{\alpha\cdot}y,\alpha\gamma)$. This shows that $(ab)f_{\alpha\beta}==(ac)f_{\alpha\gamma}$ or $ab\ \sigma\ ac$ and we have proved (14).

Conversely suppose that a congruence ϱ and a left congruence σ on D exist. By Theorem 8, D is isomorphic with a general product $D/\sigma\overline{\times}D/\varrho$, ϱ and σ naturally induce relations on $D/\sigma\overline{\times}D/\varrho$. In this sense ϱ and σ can be regarded as the relations on $D/\sigma\overline{\times}D/\varrho$. By the assumption, $(y,\beta)\sigma(y,\gamma)$ implies $(x,\alpha)(y,\beta)\sigma(x,\alpha)(y,\gamma)$, hence $x\theta_{\alpha,\beta}y=x\theta_{\alpha,\gamma}y$ which means that $x\theta_{\alpha,\beta}y$ is independent of β. Thus we have proved that $D/\sigma\overline{\times}D/\varrho$ is a left general product.

§ 6. The structure of $\mathscr{B}_E({}_a\Pi)$.

Let a be a fixed element of E, $|E|>1$, and let $\theta \in \mathscr{B}_E$. We define f_θ and g_θ by

(15) $$xf_\theta = x\theta a \quad \text{and} \quad xg_\theta = a\theta x.$$

Then f_θ and g_θ are transformations of E,

(16) $$f_{\theta\,{}_a\Pi\,\eta} = f_\theta f_\eta \quad \text{and} \quad g_{\theta\,\Pi_a\,\eta} = g_\eta g_\theta.$$

In fact, $xf_{\theta\,{}_a\Pi\,\eta} = x(\theta\,{}_a\Pi\,\eta)a = (x\theta a)\eta a = xf_\theta f_\eta$ for all $x \in E$. This proves the first relation (16); the second one can be similarly obtained. Let h be an arbitrary transformation of E. If θ is defined by

$$x\theta y = xh \quad \text{for all} \quad x, y \in E,$$

then $f_\theta = h$.

Let $\mathscr{T}_E$ denote the full transformation semigroup over E. From the above fact, it is clear that $\theta \to f_\theta$ is a homomorphism of $\mathscr{B}_E({}_a\Pi)$ onto $\mathscr{T}_E$. Likewise $\theta \to g_\theta$ is an anti-homomorphism of $\mathscr{B}_E(\Pi_a)$ onto $\mathscr{T}_E$.

Let ϱ be the congruence on $\mathscr{B}_E({}_a\Pi)$ induced by the homomorphism $\theta \to f_\theta$. In addition we define a relation σ on $\mathscr{B}_E({}_a\Pi)$ as follows:

$\theta\sigma\eta$ for $\theta, \eta \in \mathscr{B}_E$ if and only if $x\theta y = x\eta y$ for all $y \neq a$ and all x. Clearly σ is an equivalence on $\mathscr{B}_E$. Since $|E|>1$, we have $\varrho \neq \omega$, $\sigma \neq \omega$.

To prove $\varrho \cdot \sigma = \omega$, let $\theta, \eta \in \mathscr{B}_E$. We define ξ as follows:

$$x\xi y = x\theta y \ \text{ if } \ y = a, \quad \text{and} \quad x\xi y = x\eta y \ \text{ if } \ y \neq a.$$

Then $\theta\varrho\xi$ and $\xi\sigma\eta$, hence we have proved $\varrho \cdot \sigma = \omega$. By the definition of ϱ and σ, $\theta\varrho\eta$ and $\theta\sigma\eta$ imply $\theta = \eta$, that is, $\varrho \cap \sigma = \iota$. We easily see that

$$\theta\sigma\eta \quad \text{implies} \quad (\zeta\,{}_a\Pi\theta)\sigma(\zeta\,{}_a\Pi\eta) \quad \text{for all} \quad \zeta \in \mathscr{B}_E,$$

that is, σ is a left congruence. By Theorem 9, $\mathscr{B}_E(a^*)$ is isomorphic with a left general product of $\mathscr{B}_{E/\sigma}$ by $\mathscr{B}_{E/\varrho}$. For the further study of its structure, we will explain a general case as follows:

Let T be a semigroup, F be a set, $|F|=m$; let x denote a mapping of F into T: $\lambda x = \alpha_\lambda$, $\lambda \in F$, $\alpha_\lambda \in T$. The set of all mappings x of F into T is denoted by S. We define a scalar product $\beta \cdot x$ of $\beta \in T$ and $x \in S$ as follows: if $\lambda x = \alpha_\lambda$ then $\lambda(\beta \cdot x) = \beta\alpha_\lambda$. Clearly $(\beta\gamma)\cdot x = \beta\cdot(\gamma\cdot x)$. Then we define a binary operation on $G = S \times T$ as follows:

(17) $$(x, \alpha)(y, \beta) = (\alpha \cdot y, \alpha\beta).$$

G is a left general product of S by T in which $x\theta_{\alpha,\beta}y = \alpha \cdot y$. It is easy to see that G is a semigroup. The semigroup G with (17) is determined by $m = |F|$ and the semigroup T.

Definition. The semigroup G defined above is denoted by $G=\mathscr{S}\mathscr{D}_m(T)$.

Returning to $\mathscr{B}_E({}_a\Pi)$, as we mentioned, the homomorphism $\theta \to f_\theta$ is from $\mathscr{B}_E({}_a\Pi)$ onto $\mathscr{T}_E$. Further each σ-class is associated with a mapping of $E-\{a\}$ into $\mathscr{T}_E$. Accordingly we have the following theorem:

Theorem 10. *$\mathscr{B}_E({}_a\Pi)$ is isomorphic with $\mathscr{S}\mathscr{D}_m(\mathscr{T}_E)$ where $m=|E|-1$, $\mathscr{T}_E$ is the full transformation semigroup over E.*

§ 7. Sub-general product

Let U be a subset of $S\overline{\times}T$. We define a notation

$$p_{rj_T}(U)=\{\alpha\in T;\ (x,\alpha)\in U\}.$$

Definition. If U is a subgroupoid of $S\overline{\times}_\Theta T$ and if $p_{rj_T}(U)=T$, then U is called a *sub-general product* of $S\overline{\times}_\Theta T$.

Clearly U is homomorphic onto T, and if $S\overline{\times}_\Theta T$ is a semigroup, U is a subsemigroup.

As is well known, a subdirect product U of groupoids A and T is defined to be a subgroupoid U of the direct product $A\times T$ and $p_{rj_T}(U)=T$ and $p_{rj_A}(U)=A$

In this section we will prove that if a semigroup D is homomorphic onto a semigroup T, then D is isomorphic onto a sub-general product of $S\overline{\times}_\Theta T$ for some set S and some Θ, in other words, any homomorphism φ of D onto T can be extended to a h-homomorphism φ' of some semigroup D' onto T in the sense that $D\subseteqq D'$ and $\varphi'(x)=\varphi(x)$ if $x\in D$.

Proposition 11. *Let g be a homomorphism of a semigroup D onto a semigroup T. Then D is restrictedly isomorphic onto a subdirect product of D and T with respect to g and the projection of $D\times T$ onto T.*

Proof. Let $D'=\{(x,xg);\ x\in D\}$. D' is a subsemigroup of the direct product $D\times T$. We define $h:D\to D'$ by $xh=(x,xg)$. It is easy to see that h is an isomorphism of D onto D'. Let p be the projection $D\times T\to T:(x,y)p=y$. Then $g=h\cdot p$. Therefore D is restrictedly isomorphic into $D\times T$ (i.e. onto D') with respect to g and p.

Proposition 11 shows that the existence of a sub-general product $S\overline{\times}_\Theta T$ into which D can be restrictedly embedded. However the concept "direct product" has been used instead of "general product" and D has been chosen as S. Here is a question raised:

Can we choose $|S|$ as small as possible, $|S|\leqq|D|$? Theorems 12 and 13 will answer this question.

Definition (PETRICH [7]). A non-empty subset F of a semigroup D is called a *face* of D if the complement of F is an ideal of D.

Definition. Let g be a homomorphism of a semigroup D onto a semigroup T. An element α of T is called *lowly divisible* if α has a divisor β in T (i.e. $\alpha=\beta\gamma$ or $\gamma\beta$ or $\gamma\beta\delta$ for some $\gamma,\delta\in T$ with $|D_\beta|<|D_\alpha|$, $D_\beta g=\beta$, $D_\alpha g=\alpha$).

Theorem 12. *Suppose that a semigroup D is properly homomorphic, but not h-homomorphic onto a semigroup T. Let g be the homomorphism $D\to T$ and let $D=\bigcup_{\alpha\in T} D_\alpha$ be the decomposition of D induced by $g: D_\alpha g=\alpha$. Then there is a semigroup $\bar{D}$ which satisfies the following conditions:*

(a) *$\bar{D}$ is restrictedly isomorphic onto some $S_0\overline{\times}T$ with respect to g and the projection $S_0\overline{\times}T\to T$.*
(b) *D is a face of $\bar{D}$.*
(c) *Let $n=$ l.u.b. $\{|D_\alpha|;\ \alpha\in T\}$[1]. Define the cardinal number m by $m=n+1$ if n is finite and if there is a lowly divisible element α of T such that $|D_\alpha|=n$, and by $m=n$ otherwise. Then $|S_0|=m$.*

Proof. Since g is prover, $|T|>1$ and $m>1$. For each $\alpha\in T$ let D_α be a set obtained by adjoining new elements to D_α:

$$\bar{D}_\alpha=D_\alpha\cup G_\alpha,\quad D_\alpha\cap G_\alpha=\emptyset\quad\text{and}\quad G_\alpha\cap G_\beta=\emptyset\qquad(\alpha\neq\beta)$$

such that $|\bar{D}_\alpha|=m$ for all $\alpha\in T$. In detail we arrange $\{G_\alpha\}$ as follows: $G_\alpha=\emptyset$ if and only if n is finite, $|D_\alpha|=n$, and there is no lowly divisible element α in T with $|D_\alpha|=n$; if n is infinite and $n=|D_\alpha|$, then $|G_\alpha|=1$. If $G_\alpha\neq\emptyset$ we let G_α contain a special element O_α. Since g is not a h-homomorphism, $\bigcup_{\alpha\in T} G_\alpha\neq\emptyset$.

Now let $\bar{D}=\bigcup_{\alpha\in T}\bar{D}_\alpha$. We define a binary operation $(\circ)$ on $\bar{D}$ as follows:

If $a\in\bar{D}_\alpha$ and $b\in\bar{D}_\beta$, and xy denotes the product of elements x and y in D, then set

$$a\circ b=ab\quad\text{if}\quad a\in D_\alpha\quad\text{and}\quad b\in D_\beta,\quad\text{and}\quad a\circ b=0_{\alpha\beta}\quad\text{otherwise}.$$

It is easy to check that $\bar{D}$ is associative and $a\to\alpha$, $a\in\bar{D}_\alpha$, is a proper h-homomorphism of $\bar{D}$ onto T, since $|\bar{D}_\alpha|$ is constant $m>1$ and $|T|>1$.

By Theorem 6, $\bar{D}$ is isomorphic onto $S_0\overline{\times}_\Theta T$ for some set $|S_0|=m$. Let S_0 be a set with $|S_0|=m$ and 0 be a special element of S_0. Let f_α be a one-to-one mapping of S_0 onto S_α such that $0f_\alpha=0_\alpha$. Then $\Theta=\{\theta_{\alpha,\beta};\ \alpha,\beta\in T\}$ is given as follows:

$$x\theta_{\alpha,\beta}y=\begin{cases}((xf_\alpha)(yf_\beta))f_{\alpha\beta}^{-1} & \text{if}\quad xf_\alpha\in D_\alpha,\ yf_\beta\in D_\beta,\\ 0 & \text{otherwise}\end{cases}$$

[1] We assume the well-ordering principle. Since $|D_\alpha|\leqq|D|$, the least upper bound exists.

D is isomorphic into $S_0 \overline{\times}_\Theta T$ under a mapping $xf_\alpha \to (x, \alpha)$, $\alpha \in T$. It is easy to see that this is a restricted isomorphism with respect to g and the projection $S_0 \overline{\times} T \to T$. Clearly $\bigcup_{\alpha \in T} G_\alpha$ is the complement of D and is an ideal of $\bar{D}$. Therefore D is a face of D. Thus the proof of the theorem has been completed.

Theorem 13. *Suppose that a semigroup D is properly homomorphic, but not h-homomorphic onto a semigroup T. Let g be the homomorphism $D \to T$ and let $D = \bigcup_{\alpha \in T} D_\alpha$ be the decomposition of D induced by g. Then there is a semigroup $\bar{D}$ which contains D such that*

(α) *$\bar{D}$ is restrictedly isomorphic onto some $S_0 \overline{\times} T$ with respect to g and the projection $S_0 \overline{\times} T \to T$;*
(β) *$\bar{D}$ is an inflation of D* (cf. [2]);
(γ) *$|S_0| = n$ where $n =$* l.u.b. $\{|D_\alpha|;\ \alpha \in T\}$.

Proof. Let $\bar{D}_\alpha = D_\alpha \cup G_\alpha$, $|\bar{D}_\alpha| = n$, for each α where $D_\alpha \cap G_\alpha = \emptyset$ and G_α may be empty; $G_\alpha \cap G_\beta = \emptyset$ $(\alpha \neq \beta)$. Choose exactly one element from each D_α: $\{p_\alpha;\ \alpha \in T\}$, $p_\alpha \in D_\alpha$. Let $\bar{D} = \bigcup_{\alpha \in T} \bar{D}_\alpha$. We define an operation $(\circ)$ on $\bar{D}$ as follows:

If $a \in \bar{D}_\alpha$, $b \in \bar{D}_\beta$, then let

$$a \circ b = \begin{cases} ab & \text{if } a \in D_\alpha,\ b \in D_\beta, \\ ap_\beta & \text{if } a \in D_\alpha,\ b \in G_\beta, \\ p_\alpha b & \text{if } a \in G_\alpha,\ b \in D_\beta, \\ p_\alpha p_\beta & \text{if } a \in G_\alpha,\ b \in G_\beta, \end{cases}$$

where the products on the right side are in D.

It is easy to prove that $\bar{D}$ is a semigroup and an inflation of D and satisfies (α) through (γ). By the way $\theta_{\alpha,\beta}$ is given as follows: Let $x, y \in S_0$, $|S_0| = \bar{D}_\alpha$, and let $f_\alpha: S_0 \to \bar{D}_\alpha$ be a one-to-one, onto mapping.

$$x\theta_{\alpha,\beta}y = \begin{cases} ((xf_\alpha)(yf_\beta))f_{\alpha\beta}^{-1} & \text{if } xf_\alpha \in D_\alpha,\ yf_\beta \in D_\beta, \\ ((xf_\alpha)p_\beta)f_{\alpha\beta}^{-1} & \text{if } xf_\alpha \in D_\alpha,\ yf_\beta \in G_\beta, \\ (p_\alpha(yf_\beta))f_{\alpha\beta}^{-1} & \text{if } xf_\alpha \in G_\alpha,\ yf_\beta \in D_\beta, \\ (p_\alpha p_\beta)f_{\alpha\beta}^{-1} & \text{if } xf_\alpha \in G_\alpha,\ yf_\beta \in G_\beta. \end{cases}$$

Remark. $|S_0|$ in Theorem 12 is not necessarily minimum of $|S|$ for which $\bar{D}$ can be embedded into $S \overline{\times} T$ in our sense, strictly speaking, $|S_0|$ is either minimum or minimum plus one, while $|S_0|$ in Theorem 13 is certainly minimum. In Theorem 12, even if D is s-indecomposable, that is, if D has no proper semilattice homomorphic image, then $S_0 \overline{\times} T$ is not. In Theorem 13, however, if D is s-indecomposable, $S_0 \overline{\times} T$ is also; but $S_0 \overline{\times} T$ is not simple even if D is simple. On the other hand in Proposition 11, if D is simple, $D \times T$ is simple.

References

[1] R. H. Bruck, *A survey of binary systems*, Springer-Verlag (1958).

[2] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, vol. 1. Math. Surveys, No. 7, Amer. Math. Soc. (Providence, Rhode Island, 1961).

[3] M. Hall, *The theory of groups*, Macmillan Co. (New York, 1959).

[4] V. M. Hancock, *Commutative Scheier extensions of semigroups*, Dissertation, Tulane University, 1960.

[5] A. G. Kurosh, *The theory of groups*. Vol. 2, Chelsea Publ. Co (New York, 1960).

[6] K. Krohn, J. L. Rhodes and B. R. Tilson, The prime decomposition theorem of the algebraic theory of machnies, Algebraic theory of machines, languages and semigroups (Chapter 5). Edited by M. A. Arbib, Academic Press (New York, 1968).

[7] H. Neumann, *Varieties of groups*, Springer-Verlag (1967).

[8] M. Petrich, The maximal semilattice decomposition of a semigroup, *Math. Z.*, **85** (1964), 68—82.

[9] T. Tamura, Commutative nonpotent archimedean semigroup with cancellation law, *J. Gakugei Tokushima Univ.*, **8** (1957), 5—11.

[10] T. Tamura, Note on finite semigroups which satisfy certain group-like conditions, *Proc. Japan Acad.*, **36** (1960), 62—64.

[11] T. Tamura, Note on homogeneous homomorphisms, *Proc. Japan Acad.*, **44** (1968), 871—872.

[12] T. Tamura, Some contributions of computation to semigroups and groupoids, *Computational Problems in Abstract Algebra*, Pergamon Press (1970), 229—261.

UNIVERSITY OF CALIFORNIA
DAVIS, CALIFORNIA, U.S.A.

(Received June 25, 1970)