On n-permutable equational classes

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The product $\Theta \circ \Phi$ of two congruences Θ, Φ of an algebra A is defined by the following rule: $a \equiv b(\Theta \circ \Phi)$ if and only if $c \in A$ exists such that $a \equiv c(\Theta)$ and $c \equiv b(\Phi)$. Two congruences Θ_1 and Θ_2 are n-permutable if and only if $\Theta_1 \circ \Theta_2 \circ \circ \Theta_1 \circ \Theta_2 \circ \cdots = \Theta_2 \circ \Theta_1 \circ \Theta_2 \circ \Theta_1 \circ \cdots$, where on both sides there are n factors. An algebra A is n-permutable if every two congruences in A are n-permutable. We define an equational class to be n-permutable if every algebra of this class is n-permutable. It is well known, that an n-permutable equational class is $(n+1)$-permutable. In [1] G. GRÄTZER asks for examples of equational classes which show that n-permutability and $(n+1)$-permutability are not equivalent[1]). In this note we give an example with this property.

Theorem. *For every natural number $n>2$ there exists an $(n+1)$-permutable equational class $\mathscr{K}_n$ which is not n-permutable.*

Proof. Let n be a natural number. An n-Boolean algebra

$$\mathscr{B}=(B;\ \vee, \wedge, f_1(x), \ldots, f_n(x), o_0, o_1, \ldots, o_n)$$

is an algebra with two binary operations $\vee, \wedge$, n unary operations $f_1(x), \ldots, f_n(x)$ and $n+1$ nullary operations $o_0, o_1, \ldots, o_n$, such that the following conditions are satisfied:

1. $(B;\ \vee, \wedge)$ is a distributive lattice;

2. $x \vee o_n = o_n$; $x \vee o_0 = x$ for all $x \in B$;

3. $[(x \vee o_{i-1}) \wedge o_i] \vee f_i(x) = o_i$; $[(x \vee o_{i-1}) \wedge o_i] \wedge f_i(x) = o_{i-1}$.

The class of all n-Boolean algebras is denoted by $\mathscr{K}_n$. If $o_{i-1} \leqq x \leqq o_i$ then $f_i(x)$ is the relative complement from x in $[o_{i-1}, o_i]$, i.e. this interval is a Boolean lattice. A 1-Boolean algebra is a Boolean algebra. A finite chain $\mathscr{C}_n$ of $n+1$ elements is

[1]) For $n=2$ A. MITSCHKE [2] has solved this problem.

an n-Boolean algebra, if we take its elements as nullary operations: $o_0 < o_1 < o_2 < \cdots$ $\cdots < o_n$ $(o_i \in \mathscr{C}_n)$, and $f_i(x) = o_i$ if $x < o_i$, $f_i(x) = o_{i-1}$ *if* $x \geqq o_i$. The congruences of $\mathscr{C}_n$ are the lattice-congruences, i.e. $\mathscr{C}_n$ is not n-permutable. This shows that $\mathscr{K}_n$ is not n-permutable.

Let B denote an arbitary n-Boolean algebra and $x, y \in B$, $x > y$. Set $a_i = (o_i \wedge x) \vee y$. (Then is $a_0 = y$, $a_n = x$.) If Θ_1 and Θ_2 are arbitary congruences from B, such that $x \equiv y$ $(\Theta_1 \vee \Theta_2)$, then $a_{i-1} \equiv a_i (\Theta_1 \vee \Theta_2)$ $(i = 1, 2, \ldots, n)$. The interval $[a_{i-1}, a_i]$ is projective to a subinterval of $[o_{i-1}, o_i]$, i.e. $[a_{i-1}, a_i]$ is a Boolean lattice. Every Boolean lattice is 2-permutable and so for every i $(i = 1, 2, \ldots, n)$ there exists a $t_i \in [a_{i-1}, a_i]$ such that

$$a_{i-1} \equiv t_i(\Theta_1) \; i \text{ odd}, \quad a_{i-1} \equiv t_i(\Theta_2) \; i \text{ even}, \quad a_i \equiv t_i(\Theta_1) \; i \text{ even}, \; a_i \equiv t_i(\Theta_2) \; i \text{ odd}.$$

We have therefore between x, y a chain $y_0 = a_0 = y$, $y_1 = t_1$, $y_2 = t_2, \ldots, y_n = x = a_n$ with $n+1$ elements, such that $y_{i-1} \equiv y_i(\Theta_1)$ if i even and $y_{i-1} \equiv y_i(\Theta_2)$ if i odd. $\mathscr{K}_n$ is therefore $(n+1)$-permutable.

Remark. An equational class is $(n+1)$-permutable if and only if there exists $(n+2)$-ary algebraic operations $p_0, \ldots, p_{n+1}$ satisfying the following identities (see [3]):

$$p_0(x_0, \ldots, x_{n+1}) = x_0, \; p_{i-1}(x_0, x_0, x_2, x_2, \ldots) = p_i(x_0, x_0, x_2, x_2, \ldots) \; (i = \text{even}),$$

$$p_{i-1}(x_0, x_1, x_1, x_3, x_3, \ldots) = p_i(x_0, x_1, x_1, x_3, x_3, \ldots) \; (i \text{ odd}),$$

$$p_{n+1}(x_0, \ldots, x_{n+1}) = x_{n+1}.$$

A. Mitschke and H. Werner have considered for the class $\mathscr{K}_n$ the algebraic operations:

$$p_i(x_0, x_1, \ldots, x_{n+1}) = (x_i \wedge f_{n+1-i}(x_{i+1}) \vee x_{i+2}) \vee (x_{i+2} \wedge (f_i(x_{i+1}) \vee x_i))$$

which show that $\mathscr{K}_n$ is $(n+1)$-permutable.

Bibliography

[1] G. Grätzer, Two Mal'cev type theorems in universal algebra, *J. Comb. Theory*, **8** (1970), 334—342.

[2] A. Mitschke, Implication algebras are 3-permutable and 3-distributive, *Algebra Universalis*, **1** (1971), 1862—186.

[3] E. T. Schmidt, Kongruenzrelationen algebraischer Strukturen, *Math. Forschungsberichte*, **25** (Berlin, 1969).

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