On a linear transformation in the theory of probability

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1. Introduction. In the theory of random fluctuations we frequently encounter the following problem: A sequence of mutually independent and identically distributed real random variables $\{\xi_n;\ n=1, 2, \ldots\}$ is given. We define a sequence of random variables $\{\eta_n;\ n=0, 1, 2, \ldots\}$ by the recurrence formula $\eta_n = \max(0, \eta_{n-1}+\xi_n)$ $(n=1, 2, \ldots)$, where η_0 is a nonnegative random variable which is independent of the sequence $\{\xi_n\}$. The problem is to find the distribution function or the Laplace—Stieltjes transform of η_n for every $n=1, 2, \ldots$. We have several methods at our disposal for finding the generating function

$$\sum_{n=0}^{\infty} \mathrm{E}\{e^{-s\eta_n}\}\varrho^n$$

for $\mathrm{Re}(s)\geqq 0$ and $|\varrho|<1$; namely, analytical methods (F. POLLACZEK [12], [13], I. J. GOOD [6], J. H. B. KEMPERMAN [7]), algebraic methods (G. BAXTER [2], [3], J. G. WENDEL [18], [19], J. F. C. KINGMAN [8], [9], G.-C. ROTA [14]), combinatorial methods (E. S. ANDERSEN [1], F. SPITZER [16], W. FELLER [5], L. TAKÁCS [17]), and the method of factorization (see e.g. J. H. B. KEMPERMAN [7] and A. A. BOROVKOV [4]). The method of factorization has been introduced by N. WIENER and E. HOPF [21] for solving integral equations. (See also F. SMITHIES [15], H. WIDOM [20], and N. I. MUSKHELISHVILI [10].) It seems that all the existing methods have certain limitations. The analytic method of Pollaczek is constructive and gives the solution in a closed form; however, certain restrictions should be imposed on the distribution function of ξ_n. Furthermore, since the solution appears as a solution of a singular integral equation, the uniqueness of the solution should be proved. The algebraic methods are mostly descriptive, and even in the particular case when $\mathbf{P}\{\eta_0=0\}=1$, the solution does not appear in a closed form. In general, combinatorial methods do not provide the solution in a closed form either, but fortunately, in some partic-

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ular cases, we can obtain explicit expressions for $\mathbf{P}\{\eta_n \leqq x\}$ $(n=1, 2, \ldots)$. The method of factorization is mostly restricted to the case of $\mathbf{P}\{\eta_0=0\}=1$.

In what follows we shall consider a more general problem than the one mentioned above, namely, the problem of finding a sequence of functions $\Gamma_n(s)$ $(n=1, 2, \ldots)$ defined for $\mathrm{Re}(s)=0$ by a recurrence relation $\Gamma_n(s)=\mathbf{T}\{\gamma(s)\Gamma_{n-1}(s)\}$, where $\gamma(s)$ and $\Gamma_0(s)$ are elements of a commutative Banach algebra $\mathbf{R}$, $\mathbf{T}$ is a projection and $\mathbf{T}\{\Gamma_0(s)\}=\Gamma_0(s)$. We shall define $\mathbf{R}$ in such a way that on the one hand $\mathbf{R}$ is large enough to contain all the important functions arising in fluctuation theory and on the other hand $\mathbf{R}$ is small enough to allow an explicit representation of the transformation $\mathbf{T}$, which is suitable for calculations. We shall provide a constructive method for finding the generating function of $\Gamma_n(s)$ $(n=0, 1, 2, \ldots)$, and we shall obtain the solution in a closed form. As a byproduct we obtain the method of factorization and we shall show how it can be applied in the general case.

2. A Banach algebra R. Denote by $\mathbf{R}$ the space of functions $\Phi(s)$ defined for $\mathrm{Re}(s)=0$ on the complex plane, which can be represented in the form

$$\Phi(s)=\mathbf{E}\{\zeta e^{-s\eta}\}, \tag{1}$$

where ζ is a complex (or real) random variable with $\mathbf{E}\{|\zeta|\}<\infty$, and η is a real random variable. The function $\Phi(s)$ is uniquely determined by the joint distribution of ζ and η. However, there are infinitely many possible distributions which yield the same $\Phi(s)$. It follows from (1) that $|\Phi(s)|\leqq\mathbf{E}\{|\zeta|\}$ for $\mathrm{Re}(s)=0$.

Let us define the norm of $\Phi(s)$ by

$$\|\Phi\|=\inf_{\zeta}\mathbf{E}\{|\zeta|\} \tag{2}$$

where the infimum is taken for all ζ for which (1) holds (with a suitable η). Obviously, $|\Phi(s)|\leqq\|\Phi\|$ for $\mathrm{Re}(s)=0$.

We have $\|\Phi\|\geqq 0$, and $\|\Phi\|=0$ if and only if $\Phi(s)\equiv 0$. If α is a complex (or real) number and $\Phi(s)\in\mathbf{R}$, then $\alpha\Phi(s)\in\mathbf{R}$ and $\|\alpha\Phi\|=|\alpha|\,\|\Phi\|$. Furthermore, if $\Phi_1(s)\in\mathbf{R}$ and $\Phi_2(s)\in\mathbf{R}$, then $\Phi_1(s)+\Phi_2(s)\in\mathbf{R}$ and $\|\Phi_1+\Phi_2\|\leqq\|\Phi_1\|+\|\Phi_2\|$. The last statement can be proved as follows:

For any $\varepsilon>0$ let $\Phi_1(s)=\mathbf{E}\{\zeta_1 e^{-s\eta_1}\}$, where $\mathbf{E}\{|\zeta_1|\}\leqq\|\Phi_1\|+\varepsilon$, and let $\Phi_2(s)=\mathbf{E}\{\zeta_2 e^{-s\eta_2}\}$, where $\mathbf{E}\{|\zeta_2|\}\leqq\|\Phi_2\|+\varepsilon$. Let ν be a random variable which is independent of (ζ_1,η_1) and (ζ_2,η_2), and for which $\mathbf{P}\{\nu=1\}=\mathbf{P}\{\nu=2\}=\frac{1}{2}$. Let us define $\zeta=2\zeta_\nu$ and $\eta=\eta_\nu$. Then

$$\mathbf{E}\{\zeta e^{-s\eta}\}=\Phi_1(s)+\Phi_2(s) \quad\text{and}\quad \mathbf{E}\{|\zeta|\}=\mathbf{E}\{|\zeta_1|\}+\mathbf{E}\{|\zeta_2|\}<\infty. \tag{3}$$

Thus $\Phi_1(s)+\Phi_2(s)\in\mathbf{R}$, and $\|\Phi_1+\Phi_2\|\leqq\|\Phi_1\|+\|\Phi_2\|+2\varepsilon$. Since $\varepsilon>0$ is arbitrary, this proves the statement.

In what follows we shall not make use of the completeness of **R**. However, we can prove that **R** is complete, and therefore **R** is a Banach space.

Next we observe that if $\Phi_1(s)\in\mathbf{R}$ and $\Phi_2(s)\in\mathbf{R}$, then $\Phi_1(s)\Phi_2(s)\in\mathbf{R}$ and $\|\Phi_1\Phi_2\|\leqq\|\Phi_1\|\,\|\Phi_2\|$. To prove this let us define $\Phi_1(s)$ and $\Phi_2(s)$ in exactly the same way as above. However, let us assume now that (ζ_1,η_1) and (ζ_2,η_2) are independent and take $\zeta=\zeta_1\zeta_2$ and $\eta=\eta_1+\eta_2$. Then

$$\mathbf{E}\{\zeta e^{-s\eta}\}=\Phi_1(s)\Phi_2(s)\quad\text{and}\quad\mathbf{E}\{|\zeta|\}=\mathbf{E}\{|\zeta_1|\}\mathbf{E}\{|\zeta_2|\}<\infty. \tag{4}$$

Thus $\Phi_1(s)\Phi_2(s)\in\mathbf{R}$ and $\|\Phi_1\Phi_2\|\leqq(\|\Phi_1\|+\varepsilon)(\|\Phi_2\|+\varepsilon)$. Since $\varepsilon>0$ is arbitrary this proves the statement.

Accordingly, **R** is a commutative Banach algebra.

3. A linear tansformation T. Let us define a transformation **T** in **R** by

$$\mathbf{T}\{\Phi(s)\}=\Phi^+(s)=\mathbf{E}\{\zeta e^{-s\eta^+}\}, \tag{5}$$

where $\eta^+=\max(0,\eta)$. As we shall show explicity in Theorem 2, the function $\Phi^+(s)$ is independent of the particular representation (1) of $\Phi(s)$. Observe that $\Phi^+(s)$ is a regular function of s in the domain $\operatorname{Re}(s)>0$, and continuous for $\operatorname{Re}(s)\geqq 0$. Furthermore, $|\Phi^+(s)|\leqq\|\Phi\|$ for $\operatorname{Re}(s)\geqq 0$.

If α is a complex (or real) number and $\Phi(s)\in\mathbf{R}$, then $\mathbf{T}\{\alpha\Phi(s)\}=\alpha\mathbf{T}\{\Phi(s)\}$. If $\Phi_1(s)\in\mathbf{R}$ and $\Phi_2(s)\in\mathbf{R}$, then $\mathbf{T}\{\Phi_1(s)+\Phi_2(s)\}=\mathbf{T}\{\Phi_1(s)\}+\mathbf{T}\{\Phi_2(s)\}$. This follows immediately from the representation (3). Obviously, $\|\mathbf{T}\|=1$. Accordingly, **T** is a bounded linear transformation. Moreover, $\mathbf{T}^2=\mathbf{T}$, that is, **T** is a projection.

We note that if $\Phi_1(s)\in\mathbf{R}$ and $\Phi_2(s)\in\mathbf{R}$, and $\mathbf{T}\{\Phi_1(s)\}=\Phi_1(s)$ and $\mathbf{T}\{\Phi_2(s)\}=\Phi_2(s)$, then $\mathbf{T}\{\Phi_1(s)\Phi_2(s)\}=\Phi_1(s)\Phi_2(s)$. Furthermore, if $\Phi_1(s)\in\mathbf{R}$ and $\Phi_2(s)\in\mathbf{R}$, and $\mathbf{T}\{\Phi_1(s)\}=c_1$ and $\mathbf{T}\{\Phi_2(s)\}=c_2$, where c_1 and c_2 are complex (or real) constans, then $\mathbf{T}\{\Phi_1(s)\Phi_2(s)\}=c_1c_2$. These statements follow immediately from the representation (4).

4. A recurrence relation. The problem mentioned in the Introduction and many other problems in the theory of probability and stochastic processes can be reduced to the problem of finding a sequence of functions $\{\Gamma_n(s)\}$ satisfying a recurrence relation of the form

$$\Gamma_n(s)=\mathbf{T}\{\gamma(s)\Gamma_{n-1}(s)\}\ \ (n=1,2,\ldots),\quad\text{with}\quad\mathbf{T}\{\Gamma_0(s)\}=\Gamma_0(s)\quad\text{and}\quad\gamma(s)\in\mathbf{R}.$$

To solve this problem we need the following auxiliary theorem.

Lemma. *Let $\Phi_n(s)\in\mathbf{R}$ for $n=0,1,2,\ldots$ and let a_n $(n=0,1,2,\ldots)$ be complex (or real) numbers. If*

$$\sum_{n=0}^{\infty}|a_n|\,\|\Phi_n\|<\infty,$$

then

$$\Psi(s)=\sum_{n=0}^{\infty} a_n \Phi_n(s) \in \mathbf{R} \quad \text{and} \quad \mathbf{T}\{\Psi(s)\}=\sum_{n=0}^{\infty} a_n \mathbf{T}\{\Phi_n(s)\}. \tag{6}$$

Proof. If we refer to the facts that $\mathbf{R}$ is complete and $\mathbf{T}$ is continuous, then the Lemma follows immediately. However, we are not making use of the completeness of $\mathbf{R}$ and therefore a separate proof is required.

For $n=0, 1, 2, \ldots$ let $\Phi_n(s)=\mathbf{E}\{\zeta_n e^{-s\eta_n}\}$, where $\mathbf{E}\{|\zeta_n|\}\leqq 2\|\Phi_n\|$. Let ν be a discrete random variable which is independent of the sequence (ζ_n, η_n) $(n=0,1,2,\ldots)$ and which takes on nonnegative integral values with probabilities $\mathbf{P}\{\nu=n\}=p_n>0$ for $n=0, 1, 2, \ldots$. Define $\zeta=a_\nu\zeta_\nu/p_\nu$ and $\eta=\eta_\nu$. Then

$$\mathbf{E}\{\zeta e^{-s\eta}\}=\sum_{n=0}^{\infty}\mathbf{P}\{\nu=n\}\frac{a_n}{p_n}\mathbf{E}\{\zeta_n e^{-s\eta_n}\}=\sum_{n=0}^{\infty}a_n\Phi_n(s)$$

and

$$\mathbf{E}\{|\zeta|\}=\sum_{n=0}^{\infty}\mathbf{P}\{\nu=n\}\frac{|a_n|}{p_n}\mathbf{E}\{|\zeta_n|\}\leqq 2\sum_{n=0}^{\infty}|a_n|\,\|\Phi_n\|<\infty.$$

Accordingly, $\Psi(s)=\mathbf{E}\{\zeta e^{-s\eta}\}$ and $\Psi(s)\in\mathbf{R}$. Furthermore, we have

$$\mathbf{T}\{\Psi(s)\}=\mathbf{E}\{\zeta e^{-s\eta^+}\}=\sum_{n=0}^{\infty}\mathbf{P}\{\nu=n\}\frac{a_n}{p_n}\mathbf{E}\{\zeta_n e^{-s\eta_n^+}\}=\sum_{n=0}^{\infty}a_n\mathbf{T}\{\Phi_n(s)\}$$

which is in agreement with (6). This completes the proof of the Lemma.

In particular, it follows from the Lemma that if $\Phi(s)\in\mathbf{R}$, then $e^{\varrho\Phi(s)}\in\mathbf{R}$ for any ϱ, and $[1-\varrho\Phi(s)]^{-1}\in\mathbf{R}$ and $\log[1-\varrho\Phi(s)]\in\mathbf{R}$ whenever $|\varrho|\,\|\Phi\|<1$. If we form the power series expansions of these functions, then we can apply $\mathbf{T}$ term by term.

Theorem 1. *Let us suppose that $\gamma(s)\in\mathbf{R}$, $\Gamma_0(s)\in\mathbf{R}$ and $\mathbf{T}\{\Gamma_0(s)\}=\Gamma_0(s)$. Define $\Gamma_n(s)$ for $n=1, 2, \ldots$ by the recurrence relation*

$$\Gamma_n(s)=\mathbf{T}\{\gamma(s)\Gamma_{n-1}(s)\}. \tag{7}$$

If $|\varrho|\,\|\gamma\|<1$, *then*

$$\sum_{n=0}^{\infty}\Gamma_n(s)\varrho^n=e^{-\mathbf{T}\{\log[1-\varrho\gamma(s)]\}}\mathbf{T}\{\Gamma_0(s)e^{-\log[1-\varrho\gamma(s)]+\mathbf{T}\{\log[1-\varrho\gamma(s)]\}}\} \tag{8}$$

for $\mathrm{Re}(s)\geqq 0$.

Proof. Let us denote the right-hand side of (8) by $U(s,\varrho)$. Obviously, $U(s,\varrho)\in\mathbf{R}$ and $\mathbf{T}\{U(s,\varrho)\}=U(s,\varrho)$. Now we shall show that $U(s,\varrho)$ satisfies the following equation

$$U(s,\varrho)-\varrho\mathbf{T}\{\gamma(s)U(s,\varrho)\}=\Gamma_0(s). \tag{9}$$

Let us introduce the function

$$h(s)=e^{\log[1-\varrho\gamma(s)]-\mathbf{T}\{\log[1-\varrho\gamma(s)]\}}$$

for $\operatorname{Re}(s)=0$. It is obvious that $h(s)\in\mathbf{R}$, $1/h(s)\in\mathbf{R}$, and $\Gamma_0(s)/h(s)\in\mathbf{R}$. We can also see immediately that

$$\mathbf{T}\{h(s)\}=1 \tag{10}$$

and

$$\mathbf{T}\left\{\frac{\Gamma_0(s)}{h(s)}-\mathbf{T}\frac{\Gamma_0(s)}{h(s)}\right\}=0. \tag{11}$$

Now (10) and (11) imply that

$$\mathbf{T}\left\{h(s)\left[\frac{\Gamma_0(s)}{h(s)}-\mathbf{T}\frac{\Gamma_0(s)}{h(s)}\right]\right\}=0, \tag{12}$$

that is,

$$\mathbf{T}\{[1-\varrho\gamma(s)]U(s,\varrho)\}=\Gamma_0(s) \tag{13}$$

whence (9) follows.

Let us expand $U(s,\varrho)$ in a power series as follows

$$U(s,\varrho)=\sum_{n=0}^{\infty}U_n(s)\varrho^n. \tag{14}$$

This series is convergent if $|\varrho|\,\|\gamma\|<1$ and evidently $U_n(s)\in\mathbf{R}$ for $n=0,1,2,\dots$. If we put (14) into (9), then we obtain that $U_0(s)=\Gamma_0(s)$ and

$$U_n(s)=\mathbf{T}\{\gamma(s)U_{n-1}(s)\} \tag{15}$$

for $n=1,2,\dots$. Accordingly, the sequence $\{U_n(s)\}$ satisfies the same recurrence relation and the same initial condition as the sequence $\{\Gamma_n(s)\}$. Thus $U_n(s)=\Gamma_n(s)$ for $n=0,1,2,\dots$ which was to be proved.

We note that by the Lemma we have

$$\mathbf{T}\{\log[1-\varrho\gamma(s)]\}=-\sum_{n=1}^{\infty}\frac{\varrho^n}{n}\mathbf{T}\{[\gamma(s)]^n\}$$

for $|\varrho|\,\|\gamma\|<1$.

If, in particular, $\Gamma_0(s)\equiv 1$, then (8) reduces to

$$\sum_{n=0}^{\infty}\Gamma_n(s)\varrho^n=e^{-\mathbf{T}\{\log[1-\varrho\gamma(s)]\}}=\exp\left\{\sum_{n=1}^{\infty}\frac{\varrho^n}{n}\mathbf{T}\{[\gamma(s)]^n\}\right\} \tag{16}$$

where $|\varrho|\,\|\gamma\|<1$.

The usefulness of formulas (8) and (16) depends on the applicability of the transformation $\mathbf{T}$. Our next aim is to give a method for finding $\mathbf{T}\{\Phi(s)\}$ for $\Phi(s)\in\mathbf{R}$ and, in particular, for finding $\mathbf{T}\{\log[1-\varrho\gamma(s)]\}$ for $\gamma(s)\in\mathbf{R}$ and $|\varrho|\,\|\gamma\|<1$.

5. A representation of T. If we know $\Phi(s)\in\mathbf{R}$ for $\mathrm{Re}(s)=0$, then $\Phi^+(s)=\mathbf{T}\{\Phi(s)\}$ is uniquely determined for $\mathrm{Re}(s)\geqq 0$ as a function which is regular in the domain $\mathrm{Re}(s)>0$ and continuous for $\mathrm{Re}(s)\geqq 0$. We can obtain $\Phi^+(s)$ explicitly by the following theorem.

Theorem 2. *If $\Phi(s)\in\mathbf{R}$, then for* $\mathrm{Re}(s)>0$ *we have*

$$\Phi^+(s)=\frac{1}{2}\Phi(0)+\lim_{\varepsilon\to 0}\frac{s}{2\pi i}\int\limits_{L_\varepsilon}\frac{\Phi(z)}{z(s-z)}\,dz, \tag{17}$$

where the path of integration $L_\varepsilon(\varepsilon>0)$ consists of the imaginary axis from $z=-i\infty$ to $z=-i\varepsilon$ and again from $z=i\varepsilon$ to $z=i\infty$.

Proof. Let $C_\varepsilon^+(\varepsilon>0)$ be the path which consists of the imaginary axis from $z=-i\infty$ to $z=-i\varepsilon$, the semicircle $c_\varepsilon^+=\left\{z: z=\varepsilon e^{i\alpha},\ -\frac{\pi}{2}\leqq\alpha\leqq\frac{\pi}{2}\right\}$, and again the imaginary axis from $z=i\varepsilon$ to $z=i\infty$. Let $C_\varepsilon^-(\varepsilon>0)$ be the path which consists of the imaginary axis from $z=-i\infty$ to $z=-i\varepsilon$, the semicircle

$$c_\varepsilon^-=\left\{z:\ z=-\varepsilon e^{i\alpha},\ -\frac{\pi}{2}\leqq\alpha\leqq\frac{\pi}{2}\right\},$$

and again the imaginary axis from $z=i\varepsilon$ to $z=i\infty$. Let $C_\varepsilon^+(R)$ $(0<\varepsilon<R)$ be a path taken in the negative direction and containing C_ε^+ from $z=-iR$ to $z=iR$ and the semicircle $c_R^+=\left\{z:\ z=Re^{-i\alpha},\ -\frac{\pi}{2}\leqq\alpha\leqq\frac{\pi}{2}\right\}$. Let $C_\varepsilon^-(R)$ $(0<\varepsilon<R)$ be a path taken in the positive direction and containing C_ε^- from $z=-iR$ to $z=iR$ and the semicircle $c_R^-=\left\{z:\ z=-Re^{-i\alpha},\ -\frac{\pi}{2}\leqq\alpha\leqq\frac{\pi}{2}\right\}$.

Since $\Phi^+(z)$ is regular inside $C_\varepsilon^+(R)$ and continuous on the boundary, it follows by Cauchy's integral formula (see e.g. [11] p. 112) that

$$\frac{s}{2\pi i}\int\limits_{C_\varepsilon^+(R)}\frac{\Phi^+(z)}{z(s-z)}\,dz=\Phi^+(s)$$

for $0<\varepsilon<\mathrm{Re}(s)$ and $|s|<R$. Since $|\Phi^+(z)|\leqq\|\Phi\|$ for $\mathrm{Re}(z)\geqq 0$, if we let $R\to\infty$ the integral on the semicircle c_R^+ tends to 0. Hence we obtain that

$$\frac{s}{2\pi i}\int\limits_{C_\varepsilon^+}\frac{\Phi^+(z)}{z(s-z)}\,dz=\Phi^+(s) \tag{18}$$

for $0<\varepsilon<\operatorname{Re}(s)$. If $\varepsilon\to 0$, then in (18) the integral taken along the semicircle c_ε^+ tends to $\Phi^+(0)/2=\Phi(0)/2$ and thus by (18)

(19) $$\lim_{\varepsilon\to 0}\frac{s}{2\pi i}\int_{L_\varepsilon}\frac{\Phi^+(z)}{z(s-z)}\,dz+\frac{1}{2}\Phi(0)=\Phi^+(s)$$

for $\operatorname{Re}(s)>0$.

Next we observe that

(20) $$\Phi(s)-\Phi^+(s)=\mathbf{E}\{\zeta e^{s[-\eta]^+}\}-\Phi(0)$$

for $\operatorname{Re}(s)=0$. This follows from the identity $e^{-s\eta}-e^{-s\eta^+}\equiv e^{-s\eta^+}(e^{s[-\eta]^+}-1)\equiv\equiv e^{s[-\eta]^+}-1$. If we extend the definition of $\Phi(s)-\Phi^+(s)$ for $\operatorname{Re}(s)\leqq 0$ by (20), then $\Phi(s)-\Phi^+(s)$ becomes regular in the domain $\operatorname{Re}(s)<0$ and continuous for $\operatorname{Re}(s)\leqq 0$. Obviously, $|\Phi(s)-\Phi^+(s)|\leqq 2\|\Phi\|$ for $\operatorname{Re}(s)\leqq 0$. By Cauchy's integral theorem (see e.g. [11] p. 105) it follows that

$$\frac{s}{2\pi i}\int_{C_\varepsilon^-(R)}\frac{\Phi(z)-\Phi^+(z)}{z(s-z)}\,dz=0$$

for $\operatorname{Re}(s)>0$. If we let $R\to\infty$, we obtain that

(21) $$\frac{s}{2\pi i}\int_{C_\varepsilon^-}\frac{\Phi(z)-\Phi^+(z)}{z(s-z)}\,dz=0.$$

If $\varepsilon\to 0$, the part of the integral taken along the semicircle of radius ε tends to $[\Phi^+(0)-\Phi(0)]/2=0$, and thus by (21)

(22) $$\lim_{\varepsilon\to 0}\frac{s}{2\pi i}\int_{L_\varepsilon}\frac{\Phi(z)-\Phi^+(z)}{z(s-z)}\,dz=0.$$

If we add (19) and (22), we obtain (17) which was to be proved. For $\operatorname{Re}(s)=0$ the function $\Phi^+(s)$ can be obtained by continuity or by an integral representation similar to (17).

We note that if $\Phi(s)=\mathbf{E}\{\zeta e^{-s\eta}\}$ exists for some $\varepsilon>0$, that is, if $\mathbf{E}\{|\zeta e^{-\varepsilon\eta}|\}<\infty$, then

(23) $$\Phi^+(s)=\frac{s}{2\pi i}\int_{C_\varepsilon^+}\frac{\Phi(z)}{z(s-z)}\,dz$$

for $\operatorname{Re}(s)>\varepsilon>0$. For in this case (21) remains valid if C_ε^- is replaced by C_ε^+, and hence (23) follows by (18).

6. A factorization. Finally, we show that for $|\varrho|\,\|\gamma\|<1$ we can also obtain $\mathbf{T}\{\log[1-\varrho\gamma(s)]\}$ by another method, namely, by the method of factorization.

Let $\gamma(s)\in\mathbf{R}$, $|\varrho|\,\|\gamma\|<1$ and suppose that

$$1-\varrho\gamma(s) = \Gamma^{+}(s,\varrho)\Gamma^{-}(s,\varrho) \tag{24}$$

for $\mathrm{Re}(s)=0$, where $\Gamma^{+}(s,\varrho)$ and $\Gamma^{-}(s,\varrho)$ as functions of s satisfy the following requirements:

A_1: $\Gamma^{+}(s,\varrho)$ is regular in the domain $\mathrm{Re}(s)>0$,
A_2: $\Gamma^{+}(s,\varrho)$ is continuous and free from zeros in $\mathrm{Re}(s)\geqq 0$,
A_3: $\log\Gamma^{+}(s,\varrho)/s\to 0$ if $\mathrm{Re}(s)\geqq 0$ and $|s|\to\infty$,
B_1: $\Gamma^{-}(s,\varrho)$ is regular in the domain $\mathrm{Re}(s)<0$,
B_2: $\Gamma^{-}(s,\varrho)$ is continuous and free from zeros in $\mathrm{Re}(s)\leqq 0$,
B_3: $\log\Gamma^{-}(s,\varrho)/s\to 0$ if $\mathrm{Re}(s)\leqq 0$ and $|s|\to\infty$.

Such a factorization always exists. For example,

$$\Gamma^{+}(s,\varrho)=e^{\mathbf{T}\{\log[1-\varrho\gamma(s)]\}} \quad\text{and}\quad \Gamma^{-}(s,\varrho)=e^{\log[1-\varrho\gamma(s)]-\mathbf{T}\{\log[1-\varrho\gamma(s)]\}} \tag{25}$$

satisfy all the requirements. Actually, the above requirements determine $\Gamma^{+}(s,\varrho)$ and $\Gamma^{-}(s,\varrho)$ up to a factor depending only on ϱ. This is the content of the next theorem.

Theorem 3. *If* $\gamma(s)\in\mathbf{R}$, $|\varrho|\,\|\gamma\|<1$ *and*

$$1-\varrho\gamma(s) = \Gamma^{+}(s,\varrho)\Gamma^{-}(s,\varrho) \tag{26}$$

for $\mathrm{Re}(s)=0$, *where* $\Gamma^{+}(s,\varrho)$ *and* $\Gamma^{-}(s,\varrho)$ *satisfy the requirements* A_1, A_2, A_3 *and* B_1, B_2, B_3 *respectively, then*

$$\mathbf{T}\{\log[1-\varrho\gamma(s)]\} = \log\Gamma^{+}(s,\varrho)+\log\Gamma^{-}(0,\varrho) \tag{27}$$

for $\mathrm{Re}(s)\geqq 0$.

Proof. We prove (27) for $\mathrm{Re}(s)>0$; the case $\mathrm{Re}(s)=0$ then follows by continuity. Let us define the paths L_ε, C_ε^{+}, C_ε^{-}, $C_\varepsilon^{+}(R)$, $C_\varepsilon^{-}(R)$ in the same way as in the proof of Theorem 2. Then we have

$$\frac{s}{2\pi i}\int_{C_\varepsilon^{+}}\frac{\log\Gamma^{+}(z,\varrho)}{z(s-z)}\,dz = \log\Gamma^{+}(s,\varrho) \tag{28}$$

for $0<\varepsilon<\mathrm{Re}(s)$ and

$$\frac{s}{2\pi i}\int_{C_\varepsilon^{-}}\frac{\log\Gamma^{-}(z,\varrho)}{z(s-z)}\,dz = 0 \tag{29}$$

for Re $(s)>0$, Indeed, (28) and (29) follow in a similar way as (18) and (21): first we integrate along the paths $C_\varepsilon^+(R)$ and $C_\varepsilon^-(R)$, respectively, and then let $R\to\infty$. If $\varepsilon\to 0$ in (28) and (29), then we get

$$\lim_{\varepsilon\to 0}\frac{s}{2\pi i}\int_{L_\varepsilon}\frac{\log\Gamma^+(z,\varrho)}{z(s-z)}\,dz+\frac{1}{2}\log\Gamma^+(0,\varrho)=\log\Gamma^+(s,\varrho) \tag{30}$$

and

$$\lim_{\varepsilon\to 0}\frac{s}{2\pi i}\int_{L_\varepsilon}\frac{\log\Gamma^-(z,\varrho)}{z(s-z)}\,dz-\frac{1}{2}\log\Gamma^-(0,\varrho)=0 \tag{31}$$

for Re $(s)>0$. Adding (30) and (31) we obtain (27) for Re $(s)>0$. This completes the proof of the theorem.

By using (27) we can express (8) also in the form

$$\sum_{n=0}^{\infty}\Gamma_n(s)\varrho^n=\frac{1}{\Gamma^+(s,\varrho)}\mathbf{T}\left\{\frac{\Gamma_0(s)}{\Gamma^-(s,\varrho)}\right\}, \tag{32}$$

where Re $(s)\geqq 0$ and $|\varrho|\,\|\gamma\|<1$. If $\Gamma_0(s)\equiv 1$, then (8) or (32) reduces to

$$\sum_{n=0}^{\infty}\Gamma_n(s)\varrho^n=\frac{1}{\Gamma^+(s,\varrho)\Gamma^-(0,\varrho)}, \tag{33}$$

where Re $(s)\geqq 0$ and $|\varrho|\,\|\gamma\|<1$.

The above results have numerous possible applications in the theory of probability and stochastic processes. Without going into details, we mention only the solution of the problem formulated in the Introduction. If we denote by $\gamma(s)$ the Laplace—Stieltjes transform of $\mathbf{P}\{\xi_n\leqq x\}$, that is, $\gamma(s)=\mathbf{E}\{e^{-s\xi_n}\}$ for Re $(s)=0$ and $n=1,2,\dots$, and by $\Gamma_n(s)$ the Laplace—Stieltjes transform of $\mathbf{P}\{\eta_n\leqq x\}$, that is, $\Gamma_n(s)=\mathbf{E}\{e^{-s\eta_n}\}$ for Re $(s)\geqq 0$ and $n=0,1,2,\dots$, then the generating function of the sequence $\{\Gamma_n(s)\}$ is given by (8) or by (32) for $|\varrho|<1$. If, in particular, $\mathbf{P}\{\eta_0=0\}=1$, that is, $\Gamma_0(s)\equiv 1$, then

$$\sum_{n=0}^{\infty}\Gamma_n(s)\varrho^n=e^{-\mathbf{T}\{\log[1-\varrho\gamma(s)]\}}=\exp\left\{\sum_{n=1}^{\infty}\frac{\varrho^n}{n}\mathbf{T}\{[\gamma(s)]^n\}\right\} \tag{34}$$

for $|\varrho|<1$ and Re $(s)\geqq 0$. The first version of (34) is the general case of a formula of F. Pollaczek [12] and the second version can be reduced to a formula of F. Spitzer [16].

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