Remarks on endomorphism rings of torsion-free abelian groups

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1. The commutativity of the endomorphism ring

In this paper we study endomorphism rings of torsion-free abelian groups. In [2], Problem 46(a) FUCHS asks to determine all abelian groups with commutative endomorphism ring. Later FUCHS has shown the following [3]. Call a family of groups G_α $(\alpha \in I)$ a *rigid* system if Hom $(G_\alpha, G_\beta) = 0$ or a subgroup of the rationals according as $\alpha \neq \beta$ or $\alpha = \beta$. To every cardinal m, less than the first inaccessible aleph, there exists a rigid system consisting of 2^m torsion-free groups of cardinality m.

The groups in a rigid system are obviously always indecomposable and they have commutative endomorphism rings. So the question arises: if the endomorphism ring of a torsion-free abelian group G is commutative, is G then indecomposable? It is easy to construct a counter-example. Let p_1, p_2 be different primes. G_{p_1} is the group of the rationals whose denominators are powers of p_1; G_{p_2} is similar with respect to p_2. Then $\{G_{p_1}, G_{p_2}\}$ is a rigid system and $E(G) \cong E(G_{p_1}) + E(G_{p_2})$ (ring-direct sum), since G_{p_i} is a fully invariant subgroup of $G = G_{p_1} + G_{p_2}$ (direct sum) $(i = 1, 2)$. Hence $E(G)$ is commutative, but $G = G_{p_1} + G_{p_2}$ is decomposable.

Conversely, assume that G is an indecomposable group. Is $E(G)$ then a commutative ring? For well-known indecomposable groups, such as the group Z of integers, the group Q of rationals, the group $Z(p)$ of p-adic integers, any pure subgroup G of $Z(p)$, this is true. However, one can construct a counter-example as follows:

Let R be the ring of integer quaternions i.e. elements of the form $a_0 + a_1 i + a_2 j + a_3 k$ with $a_i \in Z$ $(i = 0, 1, 2, 3)$ and $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $ik = -j = -ki$, $jk = i = -kj$ with obvious addition and multiplication. R is a reduced, torsion-free ring of rank 4. By a theorem of CORNER [1] every reduced torsion-free ring A of finite rank n is isomorphic to the endomorphism ring $E(G)$ of some reduced, torsion-free group G of rank $2n$. Hence R is isomorphic to the endomorphism ring $E(G)$ of some reduced, torsion-free group G of rank 8.

Since R has no zero-divisors, the same is true for $E(G)$. Hence 0 and 1 are the only idempotents in $E(G)$. But this implies that G is indecomposable, for if $G=G_1+G_2$ for subgroups G_1, G_2, then the projections $\pi_i: G\to G_i$, $i=1, 2$, are orthogonal idempotents of $E(G)$ whose sum $\pi_1+\pi_2 = 1$. So we get either $\pi_1=1$, $\pi_2=0$ or $\pi_1=0$, $\pi_2=1$ which means either $G_2=0$ or $G_1=0$. Hence G is indecomposable, but $E(G)\cong R$ is not commutative. Thus we have to impose stronger conditions on the group G in order that its ring of endomorphisms be commutative. We recall from [4]:

Definition 1. (cf. [4], definition 2. 1) For groups G and H, we say that

(i) G is quasi-contained in H ($G\dot{\subseteqq}H$) if $nG\subseteqq H$ for some non-zero integer n;
(ii) G is quasi-equal to H ($G\doteq H$) if $G\dot{\subseteqq}H$ and $H\dot{\subseteqq}G$;
(iii) G is quasi-decomposable if there exist non-zero independent groups A and B such that $G \doteq A+B$;
(iv) G is strongly indecomposable if G is not quasi-decomposable.

Now suppose that G is a torsion-free group of rank 2. Then G is strongly indecomposable or $G = G_1+G_2$, $G_1\cong G_2$, or $G \doteq G_1+G_2$, G_i of incomparable types, or $G \doteq S+B$, type $B<$type S.

Let $E(G)$ be the ring of endomorphisms of G. Then $E(G)$ is a torsion-free ring and $QE(G)$ is the minimal Q-algebra containing $E(G)$. $QE(G)$ can be characterized as the set of linear transformation Φ of QG (minimal Q-algebra containing G) such that $n\Phi(G)\subseteqq G$ for some $n\neq 0$ in Z.

The algebra $QE(G)$ is the *ring of quasi-endomorphisms* of G and will be denoted by $\mathrm{E}(G)$. Now if G is strongly indecomposable then $\mathrm{E}(G)$ is a quadratic number field, Q, or the ring of 2×2 triangular matrices $\left\{\begin{pmatrix} a & b\\ 0 & a\end{pmatrix}\middle|\, a, b\in Q\right\}$ with equal diagonal elements. In all cases $\mathrm{E}(G)$ is commutative, hence $E(G)$, which is a subring of $\mathrm{E}(G)$, is commutative. Hence:

If G is a strongly indecomposable group of rank 2, then $E(G)$ is commutative.

Although the condition of strong indecomposability of G is sufficient for the commutativity of $E(G)$ it is not necessary, as may be seen from $G = G_1+G_2$, G_i of incomparable types (cf. first counter-example). We can extend this result to torsion-free groups of prime rank, in case G is irreducible.

Definition 2. A group G is irreducible if it has no proper non-trivial pure fully invariant subgroups (cf. [4], definition 5. 1).

Now let G be a strongly indecomposable group of prime rank. If G is irreducible, then $E(G)$ is commutative. By Corollary 5. 6 [4], $\mathrm{E}(G)=\Gamma$ is a division ring and by Theorem 5. 5, $[\Gamma:Q]=\text{rank } G=p$ (p a prime).

Now let F be the center of Γ, then $[\Gamma:Q]=[\Gamma:F][F:Q]=p$; but $[\Gamma:F]=n^2$, so $n^2|p$ which implies $n=1$, hence $\Gamma=F$ or $\mathrm{E}(G)=\Gamma$ is commutative. Then $E(G)$, as a subring of $\mathrm{E}(G)$, is commutative. For irreducible groups G of prime rank, REID [4] has shown that G is either strongly indecomposable or equal to a direct sum of isomorphic rank one groups. Hence for these groups indecomposability implies strongly indecomposability. Hence:

Theorem 1. *Let G be an irreducible, indecomposable torsion-free group of prime rank. Then $E(G)$ is commutative.*

One might ask whether strong indecomposability is always sufficient for commutativity of the endomorphism ring. The answer is no and the counter-example is again the ring R of integer quaternions. As we have seen, $R\cong E(G)$, where G is a reduced torsion-free group of rank 8. Now the ring $\mathrm{E}(G)$ of quasi-endomorphisms of G is the quaternion field F with basis 1, i, j, k over Q.

Since F is a field it is a *local ring*, that is, a ring R with identity such that $R/J(R)$ is a division ring, where $J(R)$ is the Jacobson radical of R.

By Corollary 4. 3 [4], a torsion-free group G of finite rank is strongly indecomposable if and only if $\mathrm{E}(G)$ is a local ring. Since $F=\mathrm{E}(G)$ is such a ring, it follows that G is strongly indecomposable. However, $E(G)\cong R$ is not commutative.

For the class of irreducible groups of prime rank we have seen that they are either strongly indecomposable or equal to a direct sum of isomorphic rank one groups. Now assume that G is such a group and $E(G)$ is commutative. Then the number of direct summands in a direct sum representation of G cannot be greater than one.

Hence G is strongly indecomposable or G is a rank one group. A rank one group is clearly strongly indecomposable. Hence, if we use Theorem 1, we get:

Theorem 2. *Let G be an irreducible group of prime rank. Then $E(G)$ is commutative if and only if G is strongly indecomposable.*

If we omit the condition that the rank of G should be prime, we have the following result:

Theorem 3. *Let G be an irreducible group of finite rank k, such that k is square free. Then $E(G)$ is commutative if and only if G is strongly indecomposable.*

Proof. Assume $E(G)$ is commutative, then $\mathrm{E}(G)$ is commutative. Since G is irreducible, $\mathrm{E}(G)=\Gamma_m$ where Γ is a division algebra, m is the number of strongly indecomposable summands in a quasi-decomposition of G and $m[\Gamma:Q]=\operatorname{rank} G$ [4]. Since Γ_m is commutative, it follows that $m=1$, $\mathrm{E}(G)=\Gamma$ and G is strongly indecomposable. Conversely, assume that G is strongly indecomposable. Since G is irreducible, G has a quasi-decomposition $G \doteq \sum_{i=1}^{m} G_i$ with each G_i strongly indecomposable [4]. It follows that $m=1$ and $\mathrm{E}(G)=\Gamma$ is a division ring. Moreover $[\Gamma:Q]=\text{rank } G=k$. Since the dimension of Γ over its center must be a square dividing k, this dimension is 1 and $\mathrm{E}(G)=\Gamma$ is commutative. Hence $E(G)$ is commutative. Note that Theorem 2 is a special case of Theorem 3.

From [4] we use the

Definition 3. Let G be a torsion-free group of finite rank. Let S be the pure subgroup of G generated by the collection of non-zero minimal pure fully invariant subgroups of G. We call S the pseudo-socle of G.

REID [4] has shown that $G=S$ if and only if $\mathrm{E}(G)$ is semi-simple. So we investigate the commutativity of $E(G)$ under the condition that the radical of $\mathrm{E}(G)$ is zero. First we remark that the quasidecomposition of a torsion-free group of finite rank is essentially unique i.e. if G has finite rank then any quasi-decomposition of G has only finitely many summands and if

$$\sum_{i=1}^{s} H_i \doteq G \doteq \sum_{j=1}^{t} K_j$$

with the H_i and K_j strongly indecomposable ($i=1, \dots, s$; $j=1, \dots, t$), then $s=t$ and for some permutation π of $\{1, 2, \dots, t\}$ we have K_j is quasi-isomorphic to $H_{\pi(j)}$ ($j=1, \dots, t$) [4].

Theorem 4. *Let G be a torsion-free group of finite rank with* $\mathrm{E}(G)$ *semi-simple. but not simple. Then E(G) is commutative if and only if in any quasi-decomposition of G the summands have commutative endomorphism rings.*

Proof. Assume $E(G)$ is commutative, then $\mathrm{E}(G)$ is commutative. Since $\mathrm{E}(G)$ has D.C.C. on right ideals and is semi-simple, we get $\mathrm{E}(G) \cong \Delta_1 + \dots + \Delta_m$ (direct sum), where Δ_i is a field ($i=1, \dots, m$). Identify $\mathrm{E}(G)$ with this direct sum and write $\mathrm{E}(G) = \sum_{i=1}^{m} f_i \mathrm{E}(G)$, where $\Delta_i = f_i \mathrm{E}(G)$ ($i=1, \dots, m$) and f_i induces the projection of $\mathrm{E}(G)$ onto Δ_i. To this decomposition of $\mathrm{E}(G)$ there corresponds a quasi-decomposition of $G \doteq \sum_{i=1}^{m} Gf_i$ with $\mathrm{E}(Gf_i) \cong f_i \mathrm{E}(G) f_i = \Delta_i$, so that $\mathrm{E}(Gf_i)$ is a field. Hence Gf_i is strongly indecomposable ($i=1, \dots, m$) ([4], Corollary 4. 3). Hence any quasi-decomposition of G has m strongly indecomposable summands and each of these summands has a commutative quasi-endomorphism ring and therefore a commutative endomorphism ring.

Conversely, assume that the condition for G with respect to quasi-decomposability is satisfied. Since $\mathrm{E}(G)$ has D.C.C. on right ideals and is semi-simple, it may be identified with a finite direct sum of matrix rings over division rings: $\mathrm{E}(G)=$

$= \Delta_1 + \cdots + \Delta_n$ (Wedderburn). This implies there is a set $\{e_1, \dots, e_n\}$ of non-zero mutually orthogonal idempotents of $\mathrm{E}(G)$ whose sum is the identity in $\mathrm{E}(G)$: $1 = e_1 + e_2 + \cdots + e_n$. Then there is a quasi-decomposition $G \doteq {}' \sum_{i=1}^{n} Ge_i$ of G, which corresponds to the direct decomposition of $\mathrm{E}(G)$ ([4], Theorem 3. 1). Now $\mathrm{E}(Ge_i) \cong e_i \mathrm{E}(G) e_i = \Delta_i e_i = \Delta_i$, since e_i is the unit element for Δ_i, so that Δ_i must be commutative. Hence $\mathrm{E}(G)$ is commutative and therefore $E(G)$ is commutative. This completes the proof of the theorem.

From the semi-simplicity of $\mathrm{E}(G)$ one easily derives that the components Ge_i in a quasi-decomposition of G have a semi-simple quasi-endomorphism ring $\mathrm{E}(Ge_i)$, since the radical of $e_i \mathrm{E}(G) e_i$ $(\cong \mathrm{E}(Ge_i))$ is $e_i N e_i$, where N is the radical of $\mathrm{E}(G)$. Hence Theorem 4 reduces the case of groups G of finite rank with $\mathrm{E}(G)$ semi-simple but not simple to the case of strongly indecomposable groups G of finite rank with $\mathrm{E}(G)$ semisimple but not simple.

Next assume that G is a strongly indecomposable group with semi-simple $\mathrm{E}(G)$. Then $\mathrm{E}(G)$ is a division algebra ([4], Corollary 4. 3). Now we have the following sufficient condition in order that $E(G)$ be commutative: *G has a commutative $E(G)$ if G has a non-zero minimal pure fully invariant subgroup P, whose rank k is square-free.*

(Note that the case $G = P$ or G is irreducible is contained in Theorem 3.)

Indeed, if the condition is satisfied, then rank $P = [\mathrm{E}(G) : Q] = k$, k square-free. Since the dimension of $\mathrm{E}(G)$ over its center must be a square dividing k, $\mathrm{E}(G)$ is commutative and an algebraic number field. Hence $E(G)$ is commutative.

The condition is satisfied if the rank of G is 2 or 3. If G is irreducible, $G = P$ and the rank of G is square-free. If G is not irreducible, there exists a minimal non-zero pure fully invariant subgroup P in G, distinct from G, and the rank of P is 1 or 2. Hence the condition is satisfied.

2. The Jacobson radical

All the groups G considered here are torsion-free groups of finite rank. So $\mathrm{E}(G)$ always satisfies the D.C.C. for right ideals. It is well known that under this condition G is strongly indecomposable if and only if $\mathrm{E}(G)|N$ is a division ring, where N is the Jacobson radical of $\mathrm{E}(G)$ (Corollary 4. 3, [4]), i.e. $\mathrm{E}(G)$ is a local ring.

We prove now

Theorem 5. *Let G be a torsion-free group such that $\mathrm{E}(G)$ satisfies the D.C.C. on right ideals. Then the Jacobson radical of $E(G)(=J(E(G)))$ is zero implies that the Jacobson radical of $\mathrm{E}(G)(=J(\mathrm{E}(G)))$ is zero i.e. $\mathrm{E}(G)$ is semi-simple.*

Proof. Since $\mathrm{E}(G)$ satisfies D.C.C. for right ideals, $J(\mathrm{E}(G))$ coincides with the union of all left nilpotent ideals in $\mathrm{E}(G)$ and $J(\mathrm{E}(G))$ is nil. Hence $J(\mathrm{E}(G))$ is a pure ideal in $\mathrm{E}(G)$, since the nil radical of a torsion-free ring is a pure ideal ([2], p. 271). It follows that nil radical of $E(G)=E(G)\cap$ nil radical of $\mathrm{E}(G)$, according to the correspondence between pure ideals in $E(G)$ and $\mathrm{E}(G)$. So we get nil radical of $E(G) = E(G)\cap J(\mathrm{E}(G))$ and then $E(G)\cap J(\mathrm{E}(G)) \subseteqq J(E(G))$.

Now suppose $J(E(G))=0$ and let $\varphi\in J(\mathrm{E}(G))$. Then $\varphi\in\mathrm{E}(G)$, so $\exists n\neq 0\in Z$ such that $n\varphi\in E(G)$. Also $n\varphi\in J(\mathrm{E}(G))$, hence $n\varphi\in J(\mathrm{E}(G))\cap E(G) \subseteqq J(E(G)) = 0$, so $n\varphi=0$, which implies $\varphi=0$, since $\mathrm{E}(G)$ is torsion-free. Hence $J(\mathrm{E}(G))=0$. This completes the proof of Theorem 5.

Since $\mathrm{E}(G)$ is semi-simple if and only if $G=S$, it follows immediately:

Corollary. *Let G be a torsion-free group of finite rank. If the Jacobson radical $J(E(G))$ of the endomorphism ring $E(G)$ is zero, then $G=S$.*

One may ask whether $J(E(G))=0$ is a necessary condition in order that $J(\mathrm{E}(G))=0$. This is not the case as may be seen from the following example. Let $G=Z(p)$ be the group of p-adic integers. Then $E(G)=Z(p)$ and $\mathrm{E}(G)=K(p)$, the p-adic number field. Hence $J(\mathrm{E}(G))=0$, but $J(E(G))=pZ(p)$, so $J(E(G))\neq 0$. Of course, if $E(G)$ satisfies D.C.C. on right ideals, then nil radical of $E(G)=J(E(G))= E(G)\cap J(\mathrm{E}(G))$. Hence $J(E(G))=0$ if and only if $J(\mathrm{E}(G))=0$ in this case.

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