Axiomatic characterization of $\sum$-semirings

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In memoriam A. Rényi

§ 1. Introduction

J. PŁONKA [4] introduced the concept of a sum of a join-direct system of algebras and showed that if we form a sum of a non-trivial join-direct system of algebras in an equational class the new algebra satisfies only those regular equations which are satisfied in all algebras of the direct system.

Now if we take the equational class $\mathfrak{R}$ of all associative rings and form all possible sums of join-direct systems over it, we obtain an equational class $\mathfrak{R}_\Sigma$ of additively commutative semirings. By a *semiring* $\langle R, +, \circ \rangle$ we mean a universal algebra with two associative operations $+$ and $\circ$, such that $\circ$ is distributive with respect to $+$. It is *additively commutative* if $\langle R, + \rangle$ is a commutative semigroup.

It is not true that all the additively commutative semirings can be obtained by sums of joint-direct systems over associative rings.

In this note we give a simple axiomatic characterization of those semirings R which are in $\mathfrak{R}_\Sigma$, and we call them Σ-semirings. Every Σ-semiring has a unique way of representation as a sum of join-direct system of rings.

§ 2. Basic concepts and lemmas

Let $\langle I, \leqq \rangle$ be a join-semilattice, with join denoted by $\vee$.

A system $\mathfrak{U} = \langle \langle I, \leqq \rangle, \{R_i\}_{i \in I}, \{\varphi_{ij}\}_{i \leqq j} \rangle$ is called a join-direct system of associative rings if it is a direct system of associative rings whose underlying index set is a join-semilattice and

(i) for each $i \in I$, $\langle R_i, +_i, \circ_i \rangle$ is an associative ring and $R_i \cap R_j = \emptyset$ for $i \neq j$.

(ii) If $i \leqq j$ in I, then $\varphi_{ij}: R_i \to R_j$ is a ring homomorphism, subject to the conditions: (a) $\varphi_{ii}(x) = x$ for all x in R_i, (b) $i \leqq j \leqq k$ in I, then $\varphi_{jk}\varphi_{ij} = \varphi_{ik}$.

Any join-direct system $\mathfrak{U}$ of associative rings gives us an additively commutative semiring R as follows:

Set $R=\bigcup_{i\in I} R_i$ and define $+$ and $\circ$ on R by

$$x+y = \varphi_{ik}(x)+_k\varphi_{jk}(y) \text{ and } x\circ y=\varphi_{ik}(x)\circ_k\varphi_{jk}(y) \text{ if } x\in R_i,\ y\in R_j, \text{ and } k=i\vee j.$$

Then $\langle R, +, \circ\rangle$ is an additively commutative semiring. We shall call it the *sum* of $\mathfrak{U}$ and denote it by $R=S(\mathfrak{U})$.

Now let us define a unary operation * on R by setting $^*x = -x$ if $x\in R_i$, where $-x$ is the additive inverse of x in R_i.

It can be seen that * has the following properties, for all x and y in R:

$$(1)\quad {}^*({}^*x)=x, \qquad (2)\quad x+({}^*x)+x = x, \qquad (3)\quad {}^*(x+y) = ({}^*x)+({}^*y),$$

$$(4)\quad x\circ({}^*y) = ({}^*x)\circ y = {}^*(x\circ y), \qquad (5)\quad (x+({}^*x))\circ y = x+({}^*x)+y+({}^*y).$$

Now we can state our main theorem.

Theorem 1. *A semiring $\langle R, +, \circ\rangle$ is a Σ-semiring if and only if:*

(A) *$\langle R, +, \circ\rangle$ is additively commutative and*

(B) *a unary operation $^*\colon R\to R$ can be defined satisfying the above conditions* (1)—(5).

To demonstrate it, we shall need the following

Lemma 1. *Let R be a semiring satisfying conditions* (A) *and* (B) *of the above theorem. Then we have, for all x and y in R,*

(a) $x\circ(y+({}^*y)) = (y+({}^*y))\circ x = (x+({}^*x))\circ y = y\circ(x+({}^*x))$,

(b) *if* $x+({}^*x)+y+({}^*y) = y+({}^*y)$, *then* $x\circ(y+({}^*y)) = y+({}^*y)$,

(c) *if* $x+({}^*x) = y+({}^*y)$, *then* $x\circ(y+({}^*y)) = y+({}^*y)$.

Proof. (a) follows immediately by interchanging the variables x and y in (5), using commutativity of $+$ and distributivity of $\circ$ with respect to $+$. (b) is trivial and (c) follows from (2) and (b).

Lemma 2. *Let R be a semiring satisfying the conditions* (A) *and* (B) *of the theorem. Let $E(R) = \{x+({}^*x)\mid x\in R\}$. Then $E(R)$ is the set of all additive idempotents of R, and all elements of $E(R)$ are multiplicative idempotents. Furthermore, if we define $\leqq$ on $E(R)$ by setting $a\leqq b$ if and only if $a+b = b$ for $a, b\in E(R)$, then $\langle E(R); \leqq\rangle$ is a join-semilattice.*

Proof. Let $x \in R$, then

$$\big(x+(^*x)\big)+\big(x+(^*x)\big) = \big(x+(^*x)+x\big)+(^*x) = x+(^*x) \quad \text{by (2)},$$

therefore $x+(^*x)$ is an additive idempotent.

Conversely, suppose e is an additive idempotent in R. Then by (2), $e = e+e+ +(^*e) = e+(^*e)$ is in $E(R)$.

Observe $^*\big(x+(^*x)\big) = (^*x)+\big(^*(^*x)\big) = (^*x)+x = x+(^*x)$, and by Lemma 1 (b) we have $x\circ\big(x+(^*x)\big) = x+(^*x)$, $(^*x)\circ\big(x+(^*x)\big) = x+(^*x)$. Therefore $x\circ\big(x+(^*x)\big)+(^*x)\circ\big(x+(^*x)\big) = x+(^*x)+x+(^*x)$ and then $[x+(^*x)]\circ \circ\big[x+(^*x)\big] = x+(^*x)$. Therefore $x+(^*x)$ is a multiplicative idempotent. Clearly under the relation $\leqq$, $E(R)$ becomes a partially ordered set. Let $e, f \in E(R)$. We claim that $e+f = e \vee f$. Since

$$e+(e+f) = (e+e)+f = e+f$$

we have $e \leqq e+f$. Similarly, $f \leqq e+f$. Suppose $e, f \leqq g$ in $E(R)$. Then $e+g = g$, $f+g = g$. Thus $(e+g)+(f+g) = g+g$ so $(e+f)+g = g$. Therefore, $e+f \leqq g$. This shows $e \vee f = e+f$. Hence $\langle E(R); \leqq\rangle$ is a join-semilattice.

§ 3. Proof of the theorem

The necessity of the conditions (A) and (B) was proved in § 2.

Now suppose we have a semiring R which satisfies the conditions of the theorem. Define a relation $\equiv$ on R as follows: $x \equiv y$ if and only if $x+(^*x) = y+(^*y)$.

Clearly $\equiv$ is an equivalence and therefore partitions R into disjoint classes. It is clear that each class contains one and only one element of $E(R)$. Therefore, we denote the class containing an element a of $E(R)$ by R_a. Define $+_a$ and $\circ_a$ on R_a by restricting the operations $+$ and $\circ$ of R to R_a.

We want to show that $\langle R_a, +_a, \circ_a\rangle$ is an associative ring with a as its zero.

First we show that R_a is closed under $+_a$ and $\circ_a$. Let $x, y \in R_a$, then $x+(^*x) = y+(^*y) = a$. Thus $(x+y)+\big(^*(x+y)\big) = x+(^*x)+y+(^*y) = a+a = a$, $(x\circ y)+\big(^*(x\circ y)\big) = (x\circ y)+(^*x)\circ y = \big(x+(^*x)\big)\circ y = x+(^*x)$ by Lemma 1(c). Therefore $x+y$, $x\circ y \in R_a$. Moreover, it is clear that $^*x \in R_a$.

To see that $\langle R_a, +_a\rangle$ is an abelian group with zero a, let $x \in R_a$. Then

$$x+a = x+\big(x+(^*x)\big) = x \quad \text{and} \quad x+(^*x) = a.$$

Hence $\langle R_a, +_a, \circ_a\rangle$ is an associative ring.

Now for each $a \leqq b$ in $E(R)$, define a map $\varphi_{ab}\colon R_a \to R_b$ by $\varphi_{ab}(x) = x+b$ for all x in R_a. Then

I) φ_{ab} is a ring homomorphism. Let $x, y \in R_a$. Then

$$\varphi_{ab}(x+y) = x+y+b = (x+b)+(y+b) = \varphi_{ab}(x)+_b\varphi_{ab}(y)$$

and

$$\begin{aligned}\varphi_{ab}(x)\circ_b\varphi_{ab}(y) &= (x+b)\circ(y+b) = x\circ y+b\circ y+x\circ b+b\circ b \\ &= x\circ y+b\circ y+b\circ x+b\circ b && \text{(by Lemma 1(a))} \\ &= x\circ y+b\circ(x+y)+b && \text{(by Lemma 2)} \\ &= x\circ y+b+b && \text{(by Lemma 1(b))} \\ &= x\circ y+b = \varphi_{ab}(x\circ y).\end{aligned}$$

II) $\varphi_{aa}(x) = x+a = x+x+(^*x) = x$ for all x in R_a.

III) If $a\leqq b\leqq c$ in $E(R)$, then $\varphi_{bc}\varphi_{ab} = \varphi_{ac}$ because

$\varphi_{bc}(\varphi_{ab}(x)) = \varphi_{bc}(x+b) = (x+b)+c = x+c = \varphi_{ac}(x)$ for all x in R_a.

The proof will be complete if we show that

$$R = S(\langle\langle E(R);\ \leqq\rangle,\ \{R_a\}_{a\in E(R)},\ \{\varphi_{ab}\}_{a\leqq b}\rangle).$$

Clearly $R = \bigcup_{a\in E(R)} R_a$. Define operations $\oplus$ and $\odot$ on R as follows: for x, y in R

$x\oplus y = \varphi_{ac}(x)+_c\varphi_{bc}(y)$ and $x\odot y = \varphi_{ac}(x)\circ_c\varphi_{bc}(y)$ if $x\in R_a$, $y\in R_b$, $c = a+b$.

We want to show that $\oplus = +$ and $\odot = \circ$.

Let $x, y\in R$, $x\in R_a$, $y\in R_b$, $c = a+b$. We have

$$\begin{aligned}x\oplus y &= \varphi_{ac}(x)+_c\varphi_{bc}(y) = (x+c)+(y+c) = x+y+c = \\ &= x+y+a+b = (x+a)+(y+b) = \varphi_{aa}(x)+\varphi_{bb}(y) = x+y.\end{aligned}$$

Also

$$\begin{aligned}x\odot y &= \varphi_{ac}\circ_c\varphi_{bc}(y) = (x+c)\circ(y+c) = x\circ y+c\circ y+x\circ c+c\circ c = \\ &= x\circ y+c+c+c \text{ (by Lemma1 (b))} = x\circ y+c.\end{aligned}$$

Now $x\circ y\in R_c$ for

$$\begin{aligned}x\circ y+(^*(x\circ y)) &= x\circ y+(^*x)\circ y = (x+(^*x))\circ y \\ &= (x+(^*x))+(y+(^*y)) = a+b = c && \text{(Condition (5)).}\end{aligned}$$

Therefore $x\odot y = \varphi_{cc}(x\circ y) = x\circ y$. Hence

$$R = S(\langle\langle E(R),\ \leqq\rangle,\ \{R_a\}_{a\in E(R)},\ \{\varphi_{ab}\}_{a\leqq b}\rangle).$$

Corollary. *The class of all Σ-semirings form an equational class of semirings and it includes the class of all associative rings as an equational subclass.*

§ 4. Some remarks on Σ-semirings

Remark 1. It is clear that every Σ-semiring is an additively regular semiring, i.e., a semiring such that the equation $a+x+a = a$ always has a solution (cf. [1]). However, not all additively commutative and additively regular semirings are Σ-semirings.

Consider the 3-element additively commutative and additively regular semiring R with the following tables:

$+$	a	b	c
a	a	c	c
b	c	b	c
c	c	c	c

$\circ$	a	b	c
a	b	b	b
b	b	b	b
c	b	b	b

The only possible unary operation $^*: R \to R$ which can be defined that satisfies condition (B) (1)—(4) is:

$$^*a=a, \quad ^*b=b, \quad ^*c=c.$$

However $(a+(^*a))\circ b \neq a+(^*a)+b+(^*b)$.

Remark 2. Additively regular semirings arise naturally if we consider the endomorphism semiring of a Σ-semimodule over a ring R.

By a Σ-semimodule we mean a system $\langle M, +, \{f_a\}_{a\in R}, {}^*\rangle$ where:

(1) $\langle M, +\rangle$ is a commutative semigroup,

(2) for each $a\in R$, $f_a: M\to M$ satisfies:

$$f_a(x+y) = f_a(x)+f_a(y), \quad f_{a+b}(x) = f_a(x)+f_b(x), \quad f_{a\circ b}(x) = f_a(f_b(x)),$$

(3) $^*: M\to M$ satisfies:

$$^*(^*x)=x, \quad f_r(^*x)={}^*(f_r(x)), \quad ^*(x+y) = (^*x)+(^*y),$$

$$x+{}^*x+x = x, \quad f_r(x+(^*x)) = x+(^*x).$$

The concept of Σ-semimodule is the generalization of the usual left R-module. In [3], it was shown that every Σ-semimodule M is a sum of join-direct system of R-modules, i.e. $M=S(\langle\langle E(M); \leqq\rangle, \{M_a\}_{a\in E(M)}, \{\psi_{ab}\}_{a\leqq b}\rangle)$, where $E(M)$ is the set of all idempotents of M and M_a is R-module for each $a\in E(M)$. $\psi_{ab}: M_a\to M_b$ is a module homomorphism which takes x to $x+b$ for all x in M_b.

A mapping $\varphi: M\to M$ is called an R-endomorphism of M if for $x, y\in M$ and $a\in R$ we have

$$\varphi(x+y) = \varphi(x)+\varphi(y), \quad \varphi(f_a(x))=f_a(\varphi(x)), \quad \varphi(^*x)={}^*(\varphi(x)).$$

Let $\mathrm{End}_R(M)$ denote the set of all R-endomorphisms of M.

For $\varphi, \psi \in \mathrm{End}_R(M)$ we define:

$$(\varphi+\psi)(x) = \varphi(x)+\psi(x), \quad ({}^*\varphi)(x) = {}^*(\varphi(x)), \quad (\varphi\circ\psi)(x) = \varphi(\psi(x)).$$

Then $\langle \mathrm{End}_R(M), +, \circ\rangle$ is an additively commutative semiring and * satisfies conditions (1)—(4) of Theorem 1.

Theorem 2. *Let $A=\langle \mathrm{End}_R(M), +, \circ\rangle$ be the endomorphism semiring of a Σ-semimodule M. A is a Σ-semiring if and only if M is an R-module and in this case A is a ring.*

Proof. The if part is straightforward. Suppose A is a Σ-semiring then for each $\varphi, \psi \in A$ we have $(\varphi+({}^*\varphi))\circ\psi = \varphi+({}^*\varphi)+\psi+({}^*\psi)$.

Now let $x\in M$ then we have

$$((\varphi+({}^*\varphi))\circ\psi)(x) = (\varphi+({}^*\varphi))(\psi(x)) = \varphi(\psi(x))+({}^*\varphi)(\psi(x)) =$$
$$= (\varphi\circ\psi)(x)+(\varphi\circ\psi)({}^*x) = (\varphi\circ\psi)(x+{}^*x),$$
$$(\varphi+({}^*\varphi)+\psi+({}^*\psi))(x) = \varphi(x)+({}^*\varphi)(x)+\psi(x)+({}^*\psi)(x) =$$
$$= (\varphi+\psi)(x)+(\varphi+\psi)({}^*x) = (\varphi+\psi)(x+{}^*x).$$

This implies the restrictions $\varphi+\psi|_{E(M)}$ and $\varphi\circ\psi|_{E(M)}$ are equal.

Now if $E(M)$ has more than 2 elements, say $a\leqq b$, consider the following two R-endomorphisms of M

$$\varphi_1(x)=a, \quad \varphi_2(x)=b \quad \text{for every} \quad x\in M.$$

Then $(\varphi_1+\varphi_2)(x) = a+b = b$ and $(\varphi_1\circ\varphi_2)(x) = \varphi_1(\varphi_2(x)) = \varphi_1(b) = a$ for every $x\in M$. Therefore $\varphi_1+\varphi_2 \neq \varphi_1\circ\varphi_2$ on $E(M)$: a contradiction. Thus $|E(M)|=1$ which implies that M is an R-module.

Remark 3. S. M. Yusuf [6] called an additively commutative semiring whose additive semigroup is an inverse semigroup an *additively inversive hemiring*.

If we take away (4) and (5) in condition (B) of Theorem 1, we obtain an axiomatic characterization of additively inversive hemirings. This implies immediately that the class of all additively inversive hemirings is an equational class which contains the class of Σ-semirings as an equational subclass.

Since (4) always holds in additively inversive hemiring, if we consider Σ-semirings as algebras of type $\langle 2, 2, 1\rangle$, they can be defined by the following independent axioms: 1) $\langle R, +, \circ\rangle$ is an additively commutative semiring, 2) ${}^*({}^*x)=x$, 3) ${}^*(x+y) = {}^*x+{}^*y$, 4) $x+({}^*x)+x = x$, 5) $x\circ(y+({}^*y)) = x+({}^*x)+y+({}^*y)$.

Remark 4. Let R be a $\bar{\Sigma}$-semiring. If we define a map $f: R^2 \to R$ by setting $f(x, y) = x + y + (^*y)$, then it can be checked that f is a *partition function* of R (for the terminology see [4]). By Theorem 2 of [4], it induces a sum-representation of R. This representation is essentially the same as the one we obtained in the proof of Theorem 1, and by Theorem 1 of [5] this is the only possible sum-representation of R by rings.

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Added in proof. Consider Σ-semirings as algebras of $\langle 2, 2, 1\rangle$, one can show that the lattice of equational subclasses of Σ-semirings is isomorphic to the direct product of the lattice of equational subclasses of associative rings and the two element chain. The following problem is still unsolved: what is the lattice of equational subclasses of additively inverse hemirings?

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