Generalizations of the Hardy—Littlewood inequality

By JÓZSEF NÉMETH in Szeged

1. G. H. HARDY (see for instance [3], p. 239) proved the following

Theorem A. *If* $p>1$, $a_n \geqq 0$ $(n=1,2,\dots)$ *and* $A_{1n}=\sum_{i=1}^{n} a_i$, *then*

$$\sum_{n=1}^{\infty} n^{-p} A_{1n}^p < \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p \tag{1}$$

unless all a_n *vanish. The constant is best possible.*

This result was generalized by HARDY and LITTLEWOOD [2] as follows:

Theorem B. *Suppose* $p>0$, c *is real (but not necessarily positive), and* Σa_n *is a series of positive terms. Set*

$$A_{1n}=\sum_{k=1}^{n} a_k \quad \text{and} \quad A_{n\infty}=\sum_{k=n}^{\infty} a_k.$$

If $p>1$ *we have*

$$\sum_{n=1}^{\infty} n^{-c} A_{1n}^p \leqq K \sum_{n=1}^{\infty} n^{-c}(na_n)^p \quad \text{with} \quad c>1, \text{*)} \tag{2}$$

$$\sum_{n=1}^{\infty} n^{-c} A_{n\infty}^p \leqq K \sum_{n=1}^{\infty} n^{-c}(na_n)^p \quad \text{with} \quad c<1; \tag{3}$$

and if $p<1$ *we have*

$$\sum_{n=1}^{\infty} n^{-c} A_{1n}^p \geqq K \sum_{n=1}^{\infty} n^{-c}(na_n)^p \quad \text{with} \quad c>1, \tag{4}$$

$$\sum_{n=1}^{\infty} n^{-c} A_{n\infty}^p \geqq K \sum_{n=1}^{\infty} n^{-c}(na_n)^p \quad \text{with} \quad c<1. \tag{5}$$

Theorem A was generalized by HARDY ([4], p. 273—275), and then by G. M. PETERSON and G. S. DAVIES ([7], [8]), in such a way that the arithmetic means of a_n

*) K denotes a positive absolute constant, not necessarily the same at each occurrence.

in (1) are replaced by more general sums. M. IZUMI, S. IZUMI and G. M. PETERSON ([5]) gave further generalizations, notably they proved inequalities of type

$$\sum_{n=1}^{\infty} c_{n,n} f(n) \left\{ \sum_{m=1}^{n} c_{n,m} a_m \right\}^p \leqq K \sum_{n=1}^{\infty} c_{n,n} f(n) a_n^p \tag{6}$$

under certain conditions on the matrix $(c_{m,n})$, the sequence $\{f(n)\}$, and p.

Theorem B was generalized by L. LEINDLER ([6]), who replaced in (2)—(5) the sequence $\{n^{-c}\}$ by an arbitrary sequence $\{\lambda_n\}$; for instance he proved the inequality

$$\sum_{n=1}^{\infty} \lambda_n A_{1n}^p \leqq p^p \sum_{n=1}^{\infty} \lambda_n^{1-p} \left(\sum_{m=n}^{\infty} \lambda_m \right)^p a_n^p \tag{7}$$

with $p \geqq 1$ and $\lambda_n > 0$.

In the present paper we intend to generalize and to combine these results.

2. We use the following definitions:

a) $C \in M_1$ denotes that the matrix $C = (c_{m,\nu})$ satisfies the conditions:

$$c_{m,\nu} > 0 \quad (\nu \leqq m), \qquad c_{m,\nu} = 0 \quad (\nu > m) \qquad (m, \nu = 1, 2, \ldots), \quad \text{and}$$

$$0 < \frac{c_{m,\nu}}{c_{n,\nu}} \leqq N_1 \text{*)} \qquad (0 \leqq \nu \leqq n \leqq m). \tag{8}$$

b) $C \in M_2$ denotes that $c_{m,\nu} > 0$ $(\nu \geqq m)$, $c_{m,\nu} = 0$ $(\nu < m)$ $(m, \nu = 1, 2, \ldots)$, and

$$\frac{c_{m,\nu}}{c_{n,\nu}} \geqq N_2 \qquad (0 \leqq n \leqq m \leqq \nu). \tag{9}$$

c) $C \in M_3$ denotes that $c_{\nu,m} > 0$ $(\nu \geqq m)$, and $c_{\nu,m} = 0$ $(\nu < m)$ $(\nu, m = 1, 2, \ldots)$,

$$0 < \frac{c_{\nu,m}}{c_{\nu,n}} \leqq N_3 \qquad (\nu \geqq n \geqq m \geqq 0). \tag{10}$$

d) $C \in M_4$ denotes that $c_{\nu,m} > 0$ $(\nu \leqq m)$, and $c_{\nu,m} = 0$ $(\nu > m)$ $(\nu, m = 1, 2, \ldots)$,

$$\frac{c_{\nu,m}}{c_{\nu,n}} \geqq N_4 \qquad (0 \leqq \nu \leqq m \leqq n). \tag{11}$$

3. We prove the following

Theorem. *Let $a_n \geqq 0$ and $\lambda_n > 0$ $(n = 1, 2, \ldots)$ be given, and let $C = (c_{m,k})$ be a triangular matrix.*

(a) *If $C \in M_1$ and $p \geqq 1$, then*

$$\sum_{n=1}^{\infty} \lambda_n \left(\sum_{m=1}^{n} c_{n,m} a_m \right)^p \leqq N_1^{p(p-1)} p^p \sum_{n=1}^{\infty} \lambda_n^{1-p} \left(\sum_{m=n}^{\infty} \lambda_m c_{m,n} \right)^p a_n^p. \tag{12}$$

*) N_i denote positive absolute constants $(i = 1, 2, 3, 4)$.

(b) *If* $C\in M_3$ *and* $p\geqq 1$, *then*

(13) $$\sum_{m=1}^{\infty}\lambda_m\left(\sum_{n=m}^{\infty}c_{n,m}a_n\right)^p\leqq N_3^{p(p-1)}p^p\sum_{m=1}^{\infty}\lambda_m^{1-p}\left(\sum_{n=1}^{m}\lambda_n c_{m,n}\right)^p a_m^p.$$

(c) *If* $C\in M_2$ *and* $0<p\leqq 1$, *then*

(14) $$\sum_{n=1}^{\infty}\lambda_n\left(\sum_{\nu=n}^{\infty}c_{n,\nu}a_\nu\right)^p\geqq N_2^{(1-p)p}p^p\sum_{n=1}^{\infty}\lambda_n^{1-p}\left(\sum_{k=1}^{n}c_{k,n}\lambda_k\right)^p a_n^p.$$

(d) *If* $C\in M_4$ *and* $0<p\leqq 1$, *then*

(15) $$\sum_{m=1}^{\infty}\lambda_m\left(\sum_{n=1}^{m}c_{n,m}a_n\right)^p\geqq N_4^{(1-p)p}p^p\sum_{m=1}^{\infty}\lambda_m^{1-p}\left(\sum_{n=m}^{\infty}\lambda_n c_{m,n}\right)^p a_m^p.$$

4. We remark that this theorem implies LEINDLER's theorem [6], further if $\lambda_m=c_{m,m}f_{(m)}^{1-p}$ and we write $c_{m,n}f_{(m)}$ instead of elements of the matrix C, then assertion (a) includes Theorem 3 of [5], and in the case $\lambda_n=f_{(n)}^{-p}$ and $c_{k,n}=f(k)a_{k,n}$, assertion (d) reduces to Theorem 5 of [7].

5. We require the following lemmas:

Lemma 1. ([7], Lemma 1) *If* $p>1$ *and* $z_n\geqq 0$ $(n=1,2,\ldots)$ *then*

$$\left(\sum_{k=1}^{n}z_k\right)^p\leqq p\sum_{k=1}^{n}z_k\left(\sum_{\nu=1}^{k}z_\nu\right)^{p-1}.$$

The proofs of the following lemmas are similar to that of Lemma 1.

Lemma 2. *If* $0<p<1$ *and* $z_1>0$, $z_n\geqq 0$ $(n=2,3,\ldots)$ *then*

$$\left(\sum_{k=1}^{n}z_k\right)^p\geqq p\sum_{k=1}^{n}z_k\left(\sum_{\nu=1}^{k}z_\nu\right)^{p-1}.$$

Lemma 3. *If* $0<p<1$ *and* $z_n\geqq 0$ $(n=1,2,\ldots)$ *then for every natural number* N, *for which* $z_N>0$,

$$\left(\sum_{k=n}^{N}z_k\right)^p\geqq p\sum_{k=n}^{N}z_k\left(\sum_{\nu=k}^{N}z_\nu\right)^{p-1}.$$

Lemma 4. *If* $p>1$ *and* $z_n\geqq 0$ $(n=1,2,\ldots)$ *then for every natural number* N

$$\left(\sum_{k=n}^{N}z_k\right)^p\leqq p\sum_{k=n}^{N}z_k\left(\sum_{\nu=k}^{N}z_\nu\right)^{p-1}.$$

6. Proof of Theorem. For $p=1$ the assertions are obvious; we have only to interchange the order of the summations. Further we may assume that not all a_n vanish. (Otherwise the theorem is evident.)

Proof of inequality (12). By Lemma 1 we obtain for $C=(c_{m,k})\in M_1$

$$\sum_{n=1}^{N}\lambda_n\left(\sum_{m=1}^{n}c_{n,m}a_m\right)^p\leqq p\sum_{n=1}^{N}\lambda_n\sum_{m=1}^{n}c_{n,m}a_m\left(\sum_{k=1}^{m}c_{n,k}a_k\right)^{p-1}\leqq$$

$$\leqq N_1^{p-1}p\sum_{n=1}^{N}\lambda_n\sum_{m=1}^{n}c_{n,m}a_m\left(\sum_{k=1}^{m}c_{m,k}a_k\right)^{p-1}=N_1^{p-1}p\sum_{m=1}^{N}\left(\sum_{k=1}^{m}c_{m,k}a_k\right)^{p-1}a_m\sum_{n=m}^{N}\lambda_n c_{n,m}.$$

Hence, using Hölder's inequality, we have

$$\sum_{n=1}^{N}\lambda_n\left(\sum_{m=1}^{n}c_{n,m}a_m\right)^p\leqq N_1^{p-1}p\left\{\sum_{m=1}^{N}\lambda_m\left(\sum_{k=1}^{m}c_{m,k}a_k\right)^p\right\}^{1/q}\left\{\sum_{m=1}^{N}\lambda_m^{1-p}\left(\sum_{n=m}^{N}\lambda_n c_{n,m}\right)^p a_m^p\right\}^{1/p}$$

which, by a standard computation, gives assertion (a).

Proof of inequality (13). By Lemma 4 we have for $C=(c_{m,k})\in M_3$

$$\sum_{m=1}^{N}\lambda_m\left(\sum_{n=m}^{N}c_{n,m}a_n\right)^p\leqq p\sum_{m=1}^{N}\lambda_m\sum_{n=m}^{N}c_{n,m}a_n\left(\sum_{\nu=n}^{N}c_{\nu,m}a_\nu\right)^{p-1}\leqq$$

$$\leqq N_3^{p-1}\cdot p\sum_{m=1}^{N}\lambda_m\sum_{n=m}^{N}c_{n,m}a_n\left(\sum_{\nu=n}^{N}c_{\nu,n}a_\nu\right)^{p-1}=N_3^{p-1}p\sum_{n=1}^{N}\left(\sum_{\nu=n}^{N}c_{\nu,n}a_\nu\right)^{p-1}a_n\sum_{m=1}^{n}c_{n,m}\lambda_m\leqq$$

$$\leqq N_3^{p-1}p\left\{\sum_{n=1}^{N}\lambda_n\left(\sum_{\nu=n}^{N}c_{\nu,n}a_\nu\right)^p\right\}^{1/q}\left\{\sum_{n=1}^{N}\lambda_n^{1-p}\left(\sum_{m=1}^{n}c_{n,m}\lambda_m\right)^p a_n^p\right\}^{1/p}.$$

This gives assertion (b).

Proof of inequality (14). Using Lemma 3 with an index N for which $a_N>0$ we obtain

$$\sum_{n=1}^{N}\lambda_n\left(\sum_{\nu=n}^{N}c_{n,\nu}a_\nu\right)^p\geqq p\sum_{n=1}^{N}\lambda_n\sum_{\nu=n}^{N}c_{n,\nu}a_\nu\left(\sum_{k=\nu}^{N}c_{n,k}a_k\right)^{p-1}\geqq$$

$$\geqq N_2^{1-p}\cdot p\sum_{n=1}^{N}\lambda_n\sum_{\nu=n}^{N}c_{n,\nu}a_\nu\left(\sum_{k=\nu}^{N}c_{\nu,k}a_k\right)^{p-1}=N_2^{1-p}\cdot p\sum_{\nu=1}^{N}\left(\sum_{k=\nu}^{N}c_{\nu,k}a_k\right)^{p-1}a_\nu\sum_{n=1}^{\nu}\lambda_n c_{n,\nu}.$$

Hence, using Hölder's inequality ([1], p. 19) we have

$$\sum_{n=1}^{N}\lambda_n\left(\sum_{\nu=n}^{N}c_{n,\nu}a_\nu\right)^p\geqq N_2^{1-p}p\left\{\sum_{\nu=1}^{N}\lambda_\nu\left(\sum_{k=\nu}^{N}c_{\nu,k}a_k\right)^p\right\}^{1/q}\left\{\sum_{\nu=1}^{N}\lambda_\nu^{1-p}\left(\sum_{n=1}^{\nu}\lambda_n c_{n,\nu}\right)^p a_\nu^p\right\}^{1/p}.$$

Hence we obtain (14).

Proof of inequality (15). We may assume that $a_1 \neq 0$. Using Lemma 2 we have

$$\sum_{m=1}^{N} \lambda_m \left(\sum_{n=1}^{m} c_{n,m} a_n \right)^p \geqq p \sum_{m=1}^{N} \lambda_m \sum_{n=1}^{m} c_{n,m} a_n \left(\sum_{k=1}^{n} c_{k,m} a_k \right)^{p-1} \geqq$$

$$\geqq N_4^{1-p} \cdot p \sum_{m=1}^{N} \lambda_m \sum_{n=1}^{m} c_{n,m} a_n \left(\sum_{k=1}^{n} c_{k,n} a_k \right)^{p-1} = N_4^{1-p} \cdot p \sum_{n=1}^{N} \left(\sum_{k=1}^{n} c_{k,n} a_k \right)^{p-1} a_n \sum_{m=n}^{N} \lambda_m c_{n,m} \geqq$$

$$\geqq N_4^{1-p} \cdot p \left\{ \sum_{n=1}^{N} \lambda_n \left(\sum_{k=1}^{n} c_{k,n} a_k \right)^p \right\}^{1/q} \left\{ \sum_{n=1}^{N} \lambda_n^{1-p} \left(\sum_{m=n}^{N} \lambda_m c_{n,m} \right)^p a_n^p \right\}^{1/p}.$$

Hence we get the required inequality (15), and we have completed our proof.

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