Degree of approximation by Cesàro means of Fourier—Laguerre expansions

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1. The Fourier—Laguerre expansion of a function $f(x) \in L[0, \infty]$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x), \tag{1.1}$$

where

$$\Gamma(\alpha+1)\binom{n+\alpha}{n} a_n = \int_0^{\infty} e^{-x} x^{\alpha} f(x) L_n^{(\alpha)}(x)\, dx, \tag{1.2}$$

and $L_n^{(\alpha)}(x)$ denotes the Laguerre polynomials of order $\alpha > -1$, defined by the generating function

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\omega^n = (1-\omega)^{-\alpha-1} \exp\left(-\frac{x\omega}{1-\omega}\right). \tag{1.3}$$

The nth Cesàro sum of order k of the series

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(t) \tag{1.4}$$

is, by definition, the coefficient of r^n in the expression

$$(1-r)^{-k-1} \sum_{n=0}^{\infty} L_n^{(\alpha)}(t) r^n = (1-r)^{-k-1}(1-r)^{-\alpha-1} \exp\left(-\frac{tr}{1-r}\right),$$

and is therefore equal to $L_n^{(\alpha+k+1)}(t)$.

In this paper we shall discuss the order of Cesàro means of the series (1.1) at the point $x=0$. On account of the relation $L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}$, we have

$$\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(0) = \{\Gamma(\alpha+1)\}^{-1} \sum_{n=0}^{\infty} \int_0^{\infty} e^{-t} t^{\alpha} f(t) L_n^{(\alpha)}(t)\, dt \tag{1.5}$$

(see SZEGŐ [7], p. 269). Using the Cesàro means of the series (1.4), we find that the

nth Cesàro means of order k of the series (1. 5) are given by

$$(1.6)\qquad \sigma_n^{(k)}(0)=\{A_n^{(k)}\Gamma(\alpha+1)\}^{-1}\int_0^\infty e^{-t}t^\alpha f(t)L_n^{(\alpha+k+1)}(t)\,dt,$$

where
$$A_n^{(k)}=\binom{n+k}{k}.$$

The Cesàro summability of the series (1. 5) has been studied by KOGBETLIANTZ [2] and SZEGŐ [6]. It has been shown by SZEGŐ [6] and [7], p. 270, that if $f(x)$ is continuous at $x=0$ and if

$$(1.7)\qquad \int_1^\infty e^{-x/2}x^{\alpha-k-1/3}|f(x)|\,dx<\infty,$$

then the series (1. 1) is (C, k)-summable at the point $x=0$ with the sum $f(0)$, provided that $k>\alpha+1/2$.

In Theorem I of this paper we estimate the order of Cesàro means of the series (1. 5) after replacing the continuity condition in Szegő's theorem by a much lighter condition. Similar results for Fourier-trigonometric series and for ultraspherical series on a sphere were established by OBRECHKOFF [3], [4]. In Theorem II we prove an extension of Theorem I by introducing a parameter p thus arriving at a deeper insight into the behaviour of Cesàro means. Such extensions in the case of Fourier-trigonometric series were given by WANG [8] and SUNOUCHI [5], while the author [1] has earlier studied such a problem for the ultraspherical series on a sphere.

Theorem I. *If*

$$(1.8)\qquad F(t)=\int_t^\delta \frac{|f(u)|}{u}\,du=o\left(\log\frac{1}{t}\right)$$

and

$$\int_1^\infty e^{-t/2}t^{\alpha-k-1/3}|f(t)|\,dt<\infty,$$

then

$$\sigma_n^{(k)}(0)=o(\log n),$$

provided that $k>\alpha+1/2$.

2. In the proof of the theorem we shall require the following order estimates and asymptotic values of the Laguerre functions given by SZEGŐ [7], pp. 175 and 239.

Order estimates. If α is an arbitrary real number, and c and ω are fixed positive constants, and $n\to\infty$, then.

$$(2.1)\qquad L_n^{(\alpha)}(x)=\begin{cases} x^{-\alpha/2-1/4}O(n^{\alpha/2-1/4}), & \text{if}\quad c/n\leqq x\leqq\omega,\\ O(n^\alpha), & \text{if}\quad 0\leqq x\leqq c/n.\end{cases}$$

*Asymptotic property.**) If α and λ are arbitrary real numbers, $a>0$ and $0<\eta<4$, then for $n\to\infty$

(2. 2) $$\max e^{-x/2}x^{\lambda}|L_n^{(\alpha)}(x)| \sim n^{Q},$$

where

(2. 3) $$Q=\begin{cases}\max(\lambda-1/2,\ \alpha/2-1/4) & \text{if}\quad a\leqq x\leqq(4-\eta)n,\\ \max(\lambda-1/3,\ \alpha/2-1/4) & \text{if}\quad x\geqq a,\end{cases}$$

and the maximum at the left hand member of (2, 2) is taken in the respective interval pointed out in (2. 3).

3. Proof of Theorem I. From (1. 6),

(3. 1) $$\sigma_n^{(k)}(0)=\{A_n^{(k)}\Gamma(\alpha+1)\}^{-1}\left[\int_0^{1/n}+\int_{1/n}^{1}+\int_1^{\infty}\right]=I_1+I_2+I_3.$$

Using the order estimate (2. 1) we find that**)

(3. 2)
$$I_1=O(n^{-k})\int_0^{1/n}e^{-t}t^{\alpha}|f(t)|n^{\alpha+k+1}\,dt=O(n^{\alpha+1})\int_0^{1/n}t|f(t)|\,dt=$$
$$=O(n^{\alpha+1})[t^{\alpha}F(t)]_0^{1/n}+O(n^{\alpha+1})\int_0^{1/n}t^{\alpha-1}F(t)\,dt=$$
$$=O(n^{\alpha+1})\left[t^{\alpha}o\left(t\log\frac{1}{t}\right)\right]_0^{1/n}+O(n^{\alpha+1})\int_0^{1/n}o\left(t^{\alpha}\log\frac{1}{t}\right)dt=$$
$$=o(\log n)+O(n^{\alpha+1})\left[\log\frac{1}{t}\,\frac{t^{\alpha+1}}{(\alpha+1)}+\int\frac{t^{\alpha}}{\alpha+1}\,dt\right]_0^{1/n}=$$
$$=o(\log n)+o(\log n)+o(1)=o(\log n).$$

In I_2, we make use of the first estimate of $L_n^{\alpha}(x)$ given in (2. 1) and we obtain

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*) If $b_n\neq 0$ and the sequence $\dfrac{|a_n|}{|b_n|}$ has finite positive limits of determination, we write $a_n\sim b_n$.

**) Condition (1.8) implies that
$$F(t)=\int_0^t|f(u)|\,du=o\left(t\log\frac{1}{t}\right).$$

$$I_2 = O(n^{-k}) \int_{1/n}^{1} e^{-t} t^{\alpha} |f(t)| n^{(\alpha+k+1)/2-1/4} t^{-(\alpha+k+1)/2-1/4} dt =$$

(3. 3)
$$= O[n^{-k+(\alpha+k+1)/2-1/4}] \int_{1/n}^{1} t^{\alpha/2-k/2-3/4} |f(t)| \, dt =$$

$$= O[n^{\alpha/2-k/2+1/4} n^{-\alpha/2+k/2-1/4}] \int_{1/n}^{1} \frac{|f(t)|}{t} dt = O(1)\left(\int_{1/n}^{\delta} + \int_{\delta}^{1}\right) =$$

$$= O(1) o(\log n) + O(1) \int_{\delta}^{1} \frac{|f(t)|}{t} dt = o(\log n) + O(1) = o(\log n).$$

Finally, from (2. 2) and (1. 9),

$$I_3 = O(n^{-k}) \int_{1}^{\infty} e^{-t} t^{\alpha} |f(t)| \, |L_n^{(\alpha+k+1)}(t)| \, dt =$$

(3. 4)
$$= O(n^{-k}) \int_{1}^{\infty} e^{-t/2} t^{k+1/3} |L_n^{(\alpha+k+1)}(t)| e^{-t/2} t^{\alpha-1/3-k} |f(t)| \, dt =$$

$$= O(n^{-k}) \int_{1}^{\infty} e^{-t/2} t^{\alpha-k-1/3} |f(t)| O(n^k) \, dt = O(1) = o(\log n).$$

The theorem gets proved on account of (3. 1), (3. 2), (3. 3) and (3. 4).

4. An additional parameter p, $-1 < p < \infty$, may be introduced into the theorem proved above so as to obtain a still finer result:

Theorem II. *If*

$$\int_{t}^{\delta} \frac{|f(u)|}{u} du = o\left[\left(\log \frac{1}{t}\right)^{p+1}\right] \qquad (t \to 0, \ -1 < p < \infty),$$

and if

$$\int_{1}^{\infty} e^{-t/2} t^{\alpha-k-1/3} |f(t)| \, dt < \infty,$$

then $\sigma_n^{(k)}(0) = o[(\log n)^{p+1}]$, *provided that* $k > \alpha + 1/2$.

Proof. As in the proof of Theorem I, we break the integral into $I_1+I_2+I_3$. I_3 gets disposed off exactly as before. Coming to I_1, we have

$$I_1 = O(n^{-k}) \int_0^{1/n} e^{-t} t^{\alpha} |f(t)| n^{\alpha+k+1}\, dt = O(n^{\alpha+1}) \int_0^{1/n} \frac{|f(t)|}{t} t^{1+\alpha}\, dt =$$

$$= O(n^{\alpha+1}) \left[-t^{\alpha+1} \int_t^1 \frac{|f(u)|}{u}\, du \right]_{t=0}^{t=1/n} + O(n^{\alpha+1})(\alpha+1) \int_0^{1/n} t^{\alpha} \left(\int_t^1 \frac{|f(u)|}{u}\, du \right) dt =$$

$$= o[(\log n)^{p+1}] + o(n^{\alpha+1}) \int_0^{1/n} t^{\alpha} \left(\log \frac{1}{t} \right)^{p+1} dt = o[(\log n)^{p+1}].$$

The estimate for I_2 is immediately obtained from (3. 3). This completes the proof.

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