Bi-ideals in associative rings

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Throughout this paper, by a ring A we shall mean an arbitrary associative ring. For the terminology we refer to N. JACOBSON [5], N. H. McCOY [16] and L. RÉDEI [18]. In analogy to the notion of bi-ideal in semigroups (cf. A. H. CLIFFORD and G. B. PRESTON [3] vol. I) we shall study some properties of bi-ideals in rings.

For the arbitrary subsets X and Y of a ring A by the product XY we mean the additive subgroup of the ring A which is generated by the set of all products xy, where $x \in X$, and $y \in Y$. By a bi-ideal B of a ring A we understand a subring B of A satisfying the following condition:

(1) $$BAB \subseteqq B.$$

Obviously every one-sided (left or right) ideal of A is a bi-ideal, and the intersection of a left and a right ideal of A is also a bi-ideal. We note that the bi-ideals in semigroups are special cases of the (m, n)-ideals introduced by S. LAJOS [7]. He remarked that the set of all bi-ideals of a regular ring is a multiplicative semigroup [10]. Some generalizations of biideals of rings were discussed by F. SZÁSZ [22]. The concept of the bi-ideal of semigroups was introduced by R. A. GOOD and D. R. HUGHES [4]. Interesting particular cases of bi-ideals are the quasi-ideals of O. STEINFELD [19]: A submodule Q of an associative ring A is called a quasi-ideal of A if the following condition holds:

(2) $$QA \cap AQ \subseteqq Q.$$

It is known that the product of any two quasi-ideals is a bi-ideal (cf. S. LAJOS [8]). It may be remarked that in case of regular rings the notions of bi-ideal and quasi-ideal coincide (see S. LAJOS [10]). It was shown by the first named author that there exists semigroup S containing a bi-ideal B which is not a quasi-ideal of S (see. S. LAJOS [13]).

Next we formulate some general properties of bi-ideals in rings. Then we characterize two important classes of associative rings in terms of bi-ideals.

Proposition 1. *The intersection of an arbitrary set of bi-ideals B_λ $(\lambda \in \Lambda)$ of a ring A is again a bi-ideal of A.*

Proof. Set $B = \bigcap_{\lambda \in \Lambda} B_\lambda$. Evidently B is a subring of A. From the inclusions $B_\lambda A B_\lambda \subseteqq B_\lambda$ and $B \subseteqq B_\lambda$ $(\forall \lambda \in \Lambda)$ it follows that

$$BAB \subseteqq B_\lambda A B_\lambda \subseteqq B_\lambda \qquad (\forall \lambda \in \Lambda) \tag{3}$$

and consequently we have

$$BAB \subseteqq B. \tag{4}$$

This proves Proposition 1.

Proposition 2. *The intersection of a bi-ideal B of a ring A and of a subring S of A is always a bi-ideal of the ring S.*

Proof. Let us assume that

$$C = B \cap S. \tag{5}$$

Since S is a subring and $C \subseteqq S$ we conclude

$$CSC \subseteqq SSS \subseteqq S. \tag{6}$$

On the other hand

$$CSC \subseteqq BSB \subseteqq BAB \subseteqq B, \tag{7}$$

whence $CSC \subseteqq B \cap S = C$.

Proposition 3. *For an arbitrary subset T of a ring A and for a bi-ideal B of A the products BT and TB both are bi-ideals of A.*

Proof. By $TA \subseteqq A$ and $BAB \subseteqq B$ we have

$$B(TA)B \subseteqq BAB \subseteqq B. \tag{8}$$

Moreover, we have the following monotonity property of the product defined in the introduction above:

$$X \subseteqq Y \Rightarrow XZ \subseteqq YZ \tag{9}$$

for arbitrary subsets X, Y, Z of the ring A. Then (8) and (9) imply the relation

$$(BT)A(BT) \subseteqq BT, \tag{10}$$

which together with $(BT)(BT) = (BTB)T \subseteqq (BAB)T \subseteqq BT$ means that the product BT is a bi-ideal of the ring A. The proof concerning the product TB is similar to that of BT.

In an analogy to the case of semigroups (cf. S. LAJOS [8]) we obtain the following result.

Proposition 4. *Let B be an arbitrary bi-ideal of the ring A, and C be a bi-ideal of the ring B such that $C^2 = C$. Then C is a bi-ideal of the ring A.*

Proof. The suppositions $BAB \subseteqq B$ and $CBC \subseteqq C$ imply

$$CAC = C^2AC^2 \subseteqq C(BAB)C \subseteqq CBC \subseteqq C \tag{11}$$

which proves the statement.

Proposition 5. *An arbitrary associative ring A contains no non-trivial bi-ideal if and only if A either is a zero ring of prime order or A is a division ring.*

Proof. Suppose that the ring A contains no non-trivial bi-ideals. Then clearly A contains no non-trivial right ideals, and thus A satisfies the minimum condition on right ideals. Suppose that A is not semi-simple in the sense of JACOBSON. Then A is an Artinian radical ring, which is nilpotent by a well-known result due to CH. HOPKINS (cf. N. JACOBSON [5]), and finally A is a zero ring of prime order in absence of non-trivial right ideals. On the other hand, if A is semi-simple then it is a division ring by the famous WEDDERBURN—ARTIN structure theorem (cf. JACOBSON [5] or RÉDEI [18]), which proves the "only if" part of Proposition 5.

Conversely assume that A either is a zero ring of prime order or a division ring. We shall show that A has no non-trivial bi-ideals. This assertion is trivially true for a zero ring of prime order because every additive subgroup in a zero ring is a two-sided ideal. If A is a division ring and B is a non-zero bi-ideal of A, then the condition

$$BAB \subseteqq B \tag{12}$$

implies $B = A$, because in a division ring A we have $xA = A = Ax$ for every non-zero element $x \in A$, consequently

$$BAB = B(AB) = BA = A \subseteqq B \subseteqq A. \tag{13}$$

Remark 1. An elementary and short proof of the fact that a ring A containing no non-trivial right ideals either is a zero ring of prime order or a division ring, can be found in a paper of F. SZÁSZ [20].

Proposition 6. *Let T be a non-empty subset of the ring A. Then the bi-ideal of A generated by T is of the form:*

$$T_{(1,1)} = IT + T^2 + TAT, \tag{14}$$

where I denotes the ring of rational integers.

Proof. The verification of the statement is almost trivial and we omit it.

Remark 2. By Proposition 1 the intersection of any set of bi-ideals of a ring A is also a bi-ideal of A, and thus the bi-ideal $T_{(1,1)}$ defined above evidently coincides with the intersection of all the bi-ideals of A containing T.

Remark 3. By Proposition 6 we have:

(i) The principal bi-ideal $(x)_{(1,1)}$ generated by the single element x of A can be represented as follows:

(15) $$(x)_{(1,1)} = Ix + Ix^2 + xAx.$$

(ii) In the particular case of an idempotent element e of the ring A we obtain:

(16) $$(e)_{(1,1)} = eAe.$$

(iii) For an additive subgroup T of A one has:

(17) $$T_{(1,1)} = T + T^2 + TAT.$$

(iv) If S is a subring of the ring A then

(18) $$S_{(1,1)} = S + SAS.$$

Proposition 7. *For any associative ring A denote by $\bar{A}$ the set of all additive subgroups of A, and A_1 the set of all bi-ideals of A. Then $\bar{A}$ and A_1 are semigroups under multiplication of subsets (defined in the introduction of this paper), and A_1 is a two-sided ideal of $\bar{A}$.*

Proof. The statement of this proposition is an immediate consequence of Proposition 3 and the definition given in the introduction for the multiplication of subsets.

Remark 4. The multiplicative semigroup of all non-empty subsets of an arbitrary semigroup was formerly investigated by S. Lajos [8]. He proved that the set of all bi-ideals of a semigroup is a two-sided ideal of the multiplicative semigroup of all non-empty subsets of the semigroup.

Remark 5. J. Calais [2] gave an explicite example for a semigroup having two quasi-ideals whose product fails to be a quasi-ideal. In this connection it may be remarked that one of the authors, S. Lajos [10] proved that for the case of regular rings as well as for regular semigroups the product of any two quasi-ideals is again a quasi-ideal.

For the verification of the interesting fact that every left ideal of a right ideal of an arbitrary associative ring can be represented as a right ideal of a suitable left ideal of the ring, we shall prove the following statement in analogy to a semigroup-theoretical result due to S. Lajos [7].

Theorem 1. *For an arbitrary non-empty subset B of an associative ring the following conditions are pairwise equivalent:*

(I) *B is a bi-ideal of A.*
(II) *B is a left ideal of a right ideal of A.*
(III) *B is a right ideal of a left ideal of A.*

Proof. It is enough to prove that (I) is equivalent to (II), because condition (III) is the left-right dual of (II), therefore the proof of the equivalence of (I) and (III) is similar to that of (I)$\Leftrightarrow$(II).

To show that (I) implies (II), suppose that the subset B is a bi-ideal of the ring A. Let $(B)_r$ be the right ideal of A generated by B. It will be verified that B is a left ideal of the ring $(B)_r$. Indeed, the relations $(B)_r = B + BA$ and $BAB \subseteqq B$ imply

$$(B)_r B = (B+BA)B \subseteqq B^2 + BAB \subseteqq B. \tag{19}$$

Conversely, to prove that condition (II) implies (I), assume that the subset B of A is a left ideal of a right ideal R of A. Then the inclusions

$$RA \subseteqq R, \quad RB \subseteqq B \tag{20}$$

imply

$$BAB \subseteqq (RA)B \subseteqq RB \subseteqq B, \tag{21}$$

which together with the obvious fact that B is a subring of A yields the wished assertion.

In what follows we will be concerned with different properties of bi-ideals in special classes of associative rings. Among other things the characterization of some classes of rings will be given by means of bi-ideals.

Theorem 2. *For an associative ring A the following conditions are mutually equivalent:*

(I) *A is regular.*
(II) *$L \cap R = RL$ for every left ideal L and for every right ideal R of A.*
(III) *For every pair of elements a, b of A, $(a)_r \cap (b)_l = (a)_r (b)_l$.*
(IV) *For any element a of A, $(a)_r \cap (a)_l = (a)_r (a)_l$.*
(V) *$(a)_{(1,1)} = (a)_r (a)_2$ for any element a of A.*
(VI) *$(a)_{(1,1)} = aAa$ for any element a of A.*
(VII) *$QAQ = Q$ for any quasi-ideal Q of A.*
(VIII) *$BAB = B$ for any bi-ideal B of A.*

Proof.[1]) (I)$\Leftrightarrow$(II). This was proved by L. KOVÁCS [6]. It is evident that

[1]) The equivalence of conditions (I)—(VI) in case of semigroups was proved by LAJOS [9], [11].

(II)⇒(III)⇒(IV). The implication (IV)⇒(I) was proved by F. SZÁSZ [21]. Thus we have shown the equivalence of the first four conditions.

(I)⇒(V). Assume, that A is a regular ring. Then the solvability of any equation $axa=a$ implies

$$(a)_r=(ax)_r=axA \tag{22}$$

and

$$(a)_l=(xa)_l=Axa \tag{23}$$

where $(ax)^2=ax$ and $(xa)^2=xa$. Hence

$$(a)_r(a)_l = axA\cdot Axa \subseteqq aAa \tag{24}$$

and we conclude

$$(a)_r(a)_l \subseteqq Ia+Ia^2+aAa = (a)_{(1,1)}. \tag{25}$$

Conversely, by condition (IV), it is obvious that

$$(a)_{(1,1)} \subseteqq (a)_r\cap(a)_l = (a)_r(a)_l, \tag{26}$$

Thus (I) implies (V).

To prove that (V)⇒(I), suppose that the ring A satisfies condition (V). Then we have

$$(a)_{(1,1)} = (a)_r(a)_l \tag{27}$$

for any element a in A. (27) implies

$$a\in(Ia+aA)(Ia+Aa) = Ia^2+aAa+aA^2a = Ia^2+aAa. \tag{28}$$

In other words, there exists a rational integer m and an element $b\in A$, such that

$$a = ma^2+aba = a(ma+ba). \tag{29}$$

For the element $e = ma+ba$ we obtain $a=ae$ and $e^2=e$, whence

$$a = ae^2 = a(ma+ba)^2 = a(m^2a^2+maba+mba^2+baba)\in aAa.$$

This implies (I).

It is easy to show that in case of regular rings we have

$$(a)_r(a)_l=aAa, \tag{30}$$

therefore (I)⇔(VI).

(I)⇔(VII). This has been proved by J. LUH [15].

(I)⇒(VIII). This follows at once from a result of S. LAJOS [10], Theorem 1, and from the above mentioned assertion of J. LUH.

(VIII⇒(I). If A is a ring satisfying condition (VIII), then it satisfies also (VII), which implies (I).

Therefore Theorem 2 is completely proved.

Theorem 3. *The following fifteen conditions for an associative ring are pairwise equivalent:*

(I) *A is strongly regular.*

(II) *A is a two-sided [2]) regular ring.*

(III) *A is a subcommutative [3]) regular ring.*

(IV) *$B^2 = B$ for any bi-ideal B of A.*

(V) *$Q^2 = Q$ for any quasi-ideal Q of A.*

(VI) *$RL = L \cap R \subseteqq LR$ for any left ideal L and for any right ideal R of A.*

(VII) *$L \cap R = LR$ for every left ideal L and for every right ideal R of A.*

(VIII) *$L_1 \cap L_2 = L_1 L_2$ and $R_1 \cap R_2 = R_1 R_2$ for any left ideals L_1, L_2 and for any right ideals R_1, R_2 of A.*

(IX) *$L \cap T = LT$ and $R \cap T = TR$ for every left ideal L, for every right ideal R, and for every two-sided ideal T of A.*

(X) *A is regular and it is a subdirect sum of division rings.*

(IX) *A is a regular ring with no non-zero nilpotent elements.*

(XII) *$L_1 \cap L_2 = L_1 L_2$ for any two left ideals of A.*

(XIII) *$R_1 \cap R_2 = R_1 R_2$ for any two right ideals of A.*

(XIV) *$L \cap T = LT$ for any left ideal L and for any two-sided ideal T of A.*

(XV) *$R \cap T = TR$ for any right ideal R and for any two-sided ideal T of A.*

Proof. (I)$\Leftrightarrow$(II). This was proved in [14].

(II) $\Rightarrow$ (III). Assume that A is a two-sided regular ring. Then every onesided (left or right) ideal of A is a two-sided ideal in A, consequently we have

(31) $$AxA \subseteqq xA \quad \text{and} \quad AxA \subseteqq Ax.$$

The solvability of any equation $aya = a$ ($a \in A$) implies $a \in aA$ and $a \in Aa$, for every $a \in A$, therefore by (31)

(32) $$Ax \subseteqq xA \quad \text{and} \quad xA \subseteqq Ax.$$

Thus we conclude that $xA = Ax$ for every element x in A. This exactly is the (two-sided) subcommutativity of the regular ring A.

(III) $\Rightarrow$ (II). Suppose that A is a (two-sided) subcommutative regular ring. Then every principal right ideal $(a)_r$ of A can be generated by an idempotent element e of A, that is

(33) $$(a)_r = (e)_r = eA, \quad e^2 = e.$$

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[2]) An associative ring A is said to be a two-sided (or duo) ring if every one-sided (left or right) ideal of A is a two-sided ideal (cf. e.g. Thierrin [25]).

[3]) For the definition of subcommutative ring we refer to Barbilian [1]: a ring A is called (two-sided) subcommutative if $aA = Aa$ for any $a \in A$.

From condition (III) and Theorem 2 we conclude

$$A(a)_r = A(eA) = eA^2 = eA = (a)_r, \tag{34}$$

whence $(a)_r$ is a two-sided ideal. Consequently an arbitrary right ideal R of A is also a two-sided ideal of the ring A. Similarly it can be proved that every left ideal L of A is also a two-sided ideal in A. Thus we have proved that (II)$\Leftrightarrow$(III).

(I)$\Leftrightarrow$(V). This follows from Theorem 2 of L. Kovács [6] and from authors' Theorem in [14].

Next we show that (IV)$\Leftrightarrow$(V).

The implication (IV)$\Rightarrow$(V) is evident. The converse of this statement is a consequence of the above mentioned result of L. Kovács, and Theorem 1 of S. Lajos [10].

Finally the equivalence of the conditions (VI)—(XV) with each other and with condition (I) was proved [14].

Thus Theorem 3 is proved.

It is known that every regular ring is semisimple in the sense of N. Jacobson. The following assertion characterizes the semisimple rings A in the class of rings with property:

($*$) *The lattice of all right ideals of A is a chain*[4]).

Proposition 8. *For a ring A with property ($*$) the following conditions are equivalent:*

(I) *A is semisimple.*
(II) *A is regular.*
(III) *A is strongly regular.*
(IV) *A is direct sum of division rings.*
(V) *A is a division ring.*

Proof. In what follows we assume that the ring A satisfies the condition ($*$). It is easy to see, that Proposition 8 will be proved if we demonstrate the equivalence of (I) and (V), because every class (N) of rings in Proposition 8 contains the class of rings with property ($N+I$), where $N=$ I, II, III, IV.

Suppose that A is a ring with radical $J=0$. Then the intersection of the modular maximal right ideals R_λ ($\lambda \in \Lambda$) of A is (0) by N. Jacobson [5], Chapter I. In virtue of property ($*$) and of the maximality of the right ideal R_λ we conclude $R_\lambda = 0$, whence A contains no non-trivial right ideals. Therefore A is a division ring.

Proposition 8 is completely proved.

Remark 6. A subclass of the class of rings with property ($*$) was earlier discussed by E. C. Posner [17]. Moreover, L. A. Skornjakov [24] has obtained some results concerning rings with the left-right dual of property ($*$).

4) Cf. Szász [23].

Remark 7. Let A be the ring of all matrices of type 2×2 over the field with two elements. Then A is a ring with sixteen elements having the property that $BAB = B$ holds for every bi-ideal B of A. Moreover, let B_0 be the bi-ideal generated by the element

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then we obviously have $B_0^2 = 0 \neq B_0$. Evidently A is regular, but not strongly regular and A does not satisfy condition $(*)$.

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(Received September 28, 1969)