Spectra of some Hausdorff operators

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In a recent paper [2] A. BROWN, P. HALMOS, and A. L. SHIELDS investigated the Cesàro matrix of order 1 and its continuous analogoues as operators over the Hilbert spaces l^2, $L^2[0, 1]$, and $L^2[0, \infty)$. In this paper I investigate similar properties for a class of totally regular Hausdorff matrices and their continuous analogues over the spaces l^p, $L^p[0, 1]$, and $L^p[0, \infty)$ for $p>1$.

1. Discrete methods

Let $\mu=\{\mu_n\}$ be a sequence, Δ the forward difference operator defined by $\Delta\mu_k = \mu_k-\mu_{k+1}$, $\Delta^n\mu_k=\Delta(\Delta^{n-1}\mu_k)$; $k=0, 1, 2, \ldots$; $n=1, 2, 3, \ldots$. A Hausdorff matrix H is defined by $h_{nk}=\binom{n}{k}\Delta^{n-k}\mu_k$ for $k\leqq n$, $h_{nk}=0$ for $k>n$. For a regular matrix (i.e., one that preserves limits for convergent sequences) we have the representation

$$\mu_n = \int_0^1 x^n\,dq(x) \qquad (n=0, 1, 2, \ldots),$$

where $q\in BV[0, 1]$, $q(0+) = q(0) = 0$, $q(1)=1$, and $q(u) = [q(u+0)+q(u-0)]/2$ for $0<u<1$. If in addition q is nonnegative and nondecreasing over $[0, 1]$, then H is called totally regular. For other properties of Hausdorff matrices the reader may consult [4, XI].

First we shall establish some properties for all totally regular Hausdorff matrices that are defined and bounded on l^p, and then examine some of the specific methods. Let $\|H\|_p$ denote the l^p norm of such a matrix H.

Theorem 1. *Set* $H(p) = \int_0^1 x^{-1/p}\,dq(x)$. *Then* $\|H\|_p = H(p)$.

HARDY [3] shows that $\|Hs\|^p\leqq[H(p)]^p\|s\|^p$ for any positive sequence $s=\{s_n\}\in l^p$. His result is clearly extendable to an arbitrary sequence $s\in l^p$ by observing that $\|Hs\|_p^p = \sum_{n=0}^{\infty}\left|\sum_{k=0}^{n} h_{nk}s_k\right|^p \leqq \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} h_{nk}|s_k|\right]^p$. Hence $\|H\|_p\leqq H(p)$.

To prove the converse we use the argument on pages 48 and 49 of [3] to get

$$H_n(s) > (1-\eta)^2 H(p) s_n$$

for $s_n = (n+1)^{-\omega}$, $\omega = \frac{1}{p}+\varepsilon$, $0 < \varepsilon < \frac{1}{p}$, $\eta>0$ and arbitrary. This result leads to $\|H\|_p \geqq H(p)$.

Some of the well-known Hausdorff matrices which are bounded operators over l^p are the Cesàro, Hölder, Euler, gamma, and generalized-Cesàro. These are listed below along with their generating sequences and mass functions.

$$C_\alpha\colon \mu_n = \frac{\Gamma(\alpha+1)\Gamma(n+1)}{\Gamma(n+\alpha+1)}; \quad q(x) = 1-(1-x)^\alpha;$$

$$H_\alpha\colon \mu_n = (n+1)^{-\alpha}; \quad q(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (\log(1/t))^{\alpha-1}\,dt;$$

$$(E, r)\colon \mu_n = a^n,\ r = (1-a)/a; \quad q(x) = \begin{cases} 0, & 0\leqq x<a<1 \\ 1, & a\leqq x\leqq 1; \end{cases}$$

$$\Gamma_a^\alpha\colon \mu_n = \left(\frac{a}{n+a}\right)^\alpha; \quad q(x) = \frac{a^\alpha}{\Gamma(\alpha)} \int_0^x t^{a-1}(\log(1/t))^{\alpha-1}\,dt;$$

$$C_a^\alpha\colon \mu_n = \frac{\Gamma(a+\alpha)\Gamma(n+a)}{\Gamma(a)\Gamma(n+a+\alpha)}; \quad q(x) = \frac{\Gamma(a+\alpha)}{\Gamma(a)\Gamma(\alpha)} \int_0^x t^{a-1}(1-t)^{\alpha-1}\,dt.$$

From Theorem 1, with q satisfying $1/p+1/q = 1$, the corresponding l^p-norms are:

$$\|C_\alpha\|_p = \frac{\Gamma(1+\alpha)\Gamma(1/q)}{\Gamma(\alpha+1/q)}; \quad \|H_\alpha\|_p = q^\alpha; \quad \|(E, r)\|_p = (1+r)^{1/p};$$

$$\|\Gamma_a^\alpha\|_p = \left(\frac{a}{a-1/p}\right)^\alpha \quad \|C_a^\alpha\|_p = \frac{\Gamma(a+\alpha)\Gamma(a-1/p)}{\Gamma(a+\alpha-1/p)}.$$

From the above it is clear that the operators are bounded for $\alpha>0$, $r>0$, $a > 1/p$.

We shall now show that Γ_a^α is not bounded for $0 < a \leqq 1/p$. If (h_{nk}) denotes the corresponding matrix, then $\Gamma_a^\alpha(e_0) = \{h_{n0}\}$, where $h_{n0} = \frac{\Gamma(a+1)\Gamma(n+1)}{\Gamma(n+a+1)}$, and

$$\|\Gamma_a^\alpha\|_p^p \geqq \|\Gamma_a^\alpha(e_0)\|_p^p = \sum_{n=0}^\infty (h_{n0})^p = \Gamma^p(a+1) \sum_{n=0}^\infty \left(\frac{\Gamma(n+1)}{\Gamma(n+a+1)}\right)^p.$$

Using the Gauss test one can show that the series diverges for $0 < a \leqq 1/p$. For the C_a^α matrix,

$$h_{n0} = \frac{\Gamma(n+\alpha)\Gamma(a+\alpha)}{\Gamma(\alpha)\Gamma(n+a+\alpha)},$$

and, as above, it can be shown that C_a^a is not bounded for $0 < a \leqq 1/p$.

Theorem 2. *If $H \neq I$, the point spectrum of H is empty.*

Proof. Suppose $Hf = \lambda f$ for some λ. Since H is totally regular, $\mu_n \neq 0$ for each n. Thus H is not a left zero divisor in l^p and $\lambda = 0$ is not possible. Define $g(n) = \sum_{k=0}^{n} h_{nk} f(k)$ for $f \in l^p$. Since $g(0) = \lambda f(0)$, we must have $\lambda = 1$ for any f with $f(0) \neq 0$.

Case I. Assume $f(0) \neq 0$. Then $\lambda = 1$ and we have $(H-I)f \equiv 0$. In particular $h_{10}f(0) + (h_{11}-1)f(1) = 0$; i.e.,

$$f(1) = \frac{h_{10}f(0)}{1-h_{11}}.$$

But $h_{10} = \binom{1}{0}\Delta\mu_0 = \mu_0 - \mu_1 = 1 - h_{11} > 0$. Therefore $f(1) = f(0)$ and, by induction, $f(n) = f(0)$, $n = 1, 2, \dots$. Since $f(0) \neq 0$, $f = \{f(0)\} \notin l^p$.

Case II. Assume $f(0) = 0$. Then either there exists an integer N such that $\mu_N = \lambda$ or else $\mu_n \neq \lambda$ for any n.

Case IIA. $f(0) = 0$ and $\mu_n \neq \lambda$ for any n. From the equation $h_{10}f(0) + h_{11}f(1) = \lambda f(1)$ we get $(\lambda - \mu_1)f(1) = 0$ which implies $f(1) = 0$. By induction, $f(n) = 0$, $n = 0, 1, 2, \dots$.

Case IIB. $f(0) = 0$ and $\mu_N = \lambda$ for some N. If $N = 0$, then $\lambda = 1$ and we must have $h_{10}f(0) = h_{11}f(1) = f(1)$; i.e., $(1-\mu_1)f(1) = 0$. Since $H \neq I$, $\mu_1 \neq 1$. Therefore $f(1) = 0$ and by induction, $f(n) = 0$ for $n = 2, 3, 4, \dots$.

Since $\mu_n > \mu_{n+1}$ for each r (a well-known property for totally regular Hausdorff matrices $H \neq I$), if $N > 0$ then clearly $f(0) = 0$ implies $f(1) = f(2) = \dots = f(N-1) = 0$ and $f(N)$ remains undetermined.

If $f(N) = 0$, then, as before, $f \equiv 0$.

If $f(N) \neq 0$ we shall show by induction that

$$f(N+r) = \binom{N+r}{N} f(N), \qquad r = 0, 1, 2, \dots.$$

This is trivially true for $r = 0$. Assume the induction hypothesis. Then

$$\sum_{k=0}^{N+r+1} h_{N+r+1,k} f(k) = \mu_N f(N+r+1)$$

or

$$(\mu_N-\mu_{N+r+1})f(N+r+1) = \sum_{k=N}^{N+r} h_{N+r+1,k}f(k) = \sum_{j=0}^{r} h_{N+r+1,N+j}f(N+j) =$$

$$= \sum_{j=0}^{r} \binom{N+r+1}{N+j}(\Delta^{r+1-j}\mu_{N+j})\binom{N+j}{N}f(N) =$$

$$= f(N)\binom{N+r+1}{N}\sum_{j=0}^{r}\binom{r+1}{j}\Delta^{r+1-j}\mu_{N+j}.$$

Note that $\sum_{j=0}^{r+1}\binom{r+1}{j}\Delta^{r+1-j}\mu_{N+j}$ is row $(r+1)$ of the Hausdorff matrix with generating sequence $\{\mu_{N+r}\}_{r=0}^{\infty}$.

Therefore

$$\sum_{j=0}^{r}\binom{r+1}{j}\Delta^{r+1-j}\mu_{N+j} = \mu_N-\mu_{N+r+1} \neq 0,$$

and we get $f(N+r+1) = \binom{N+r+1}{N}f(N)$. Moreover, $|f(N+r+1)| > |f(N+r)|$ so that $f \notin l^p$.

Theorem 3. *For $H \neq I$, H^*-N has a total set of proper vectors corresponding to proper values of modulus strictly less than N.*

Proof. Define a family of sequences $\beta_0, \beta_1, \beta_2, \dots$ with $\beta_n = \Delta^n e_0$, where $\Delta e_0 = e_0 - e_1$, $\Delta^n e_0 = \Delta(\Delta^{n-1}e_0)$. The set $\{\beta_0, \beta_1, \dots\}$ is total over l^q, q the conjugate index of p. For $m>n$, $H^*\beta_n(m)=0$. For $m \leqq n$

$$H^*\beta_n(m) = \sum_{k=m}^{\infty} h^*_{mk}\beta_n(k) = \sum_{k=m}^{n} h^*_{mk}(-1)^k\binom{n}{k} =$$

$$= \sum_{k=m}^{n}\binom{k}{m}(\Delta^{k-m}\mu_m)(-1)^k\binom{n}{k} = \sum_{k=m}^{n}\binom{k}{m}\binom{n}{k}(-1)^k\sum_{j=0}^{k-m}(-1)^j\binom{k-m}{j}\mu_{m+j} =$$

$$= \sum_{r=0}^{n-m}\binom{r+m}{m}\binom{n}{r+m}(-1)^{r+m}\sum_{j=0}^{r}(-1)^j\binom{r}{j}\mu_{m+j} =$$

$$= \binom{n}{m}\sum_{j=0}^{n-m}(-1)^{m+j}\mu_{m+j}\sum_{r=j}^{n-m}(-1)^r\frac{(n-m)!}{(n-m-r)!\,j!\,(r-j)!} =$$

$$= (-1)^m\binom{n}{m}\sum_{j=0}^{n-m}\binom{n-m}{j}\mu_{m+j}\sum_{s=0}^{n-m-j}(-1)^s\binom{n-m-j}{s} = (-1)^m\binom{n}{m}\mu_n = \mu_n\beta_n(m).$$

Therefore $(H^*-N)\beta_n = (\mu_n-N)\beta_n$.

Based on the knownledge of the spectrum of C_1 for $p=2$, and Theorem 3, one might conjecture that $\sigma(H)$ is the disk $\{\lambda: |\lambda-N| \leqq N\}$. Unfortunately, the conjecture is false, as the following example indicates.

From [2], $\sigma(C_1)=\{\lambda: |\lambda-1| \leqq 1\}$. Let $z = 1+e^{i\theta}$. Then $z^2 = 2(1+\cos\theta)e^{i\theta}$. Let $w = x+iy = z^2$. Then putting w in polar form yields the cardioid $r = 2(1+\cos\theta)$. Since $\sigma(C_1^2)=(\sigma(C_1))^2$, $\sigma(C_1^2)$ is the closed bounded region with the above cardioid as boundary.

There is, however, a class of totally regular Hausdorff methods H for which $\sigma(H) = \{\lambda:|\lambda-N| \leqq N\}$. This class includes the gamma methods of order 1.

Theorem 4. $\sigma(\Gamma_a^1) = \{\lambda:|\lambda-N| \leqq N\}$.

The operator $N-\Gamma_a^1-\lambda$ has moment generating sequence

$$\mu_n = c-\frac{a}{n+a},$$

where $c = N-\lambda$. Let $\varepsilon_n = 1/\mu_n$. If it can be shown that H_ε is a bounded operator over l^p for $|\lambda|>N$, then $\sigma(N-\Gamma_a^1) \subseteqq \{\lambda:|\lambda| \leqq N\}$. Hence $\sigma(\Gamma_a^1) \subseteqq \{\lambda: |\lambda-N| \leqq N\}$. Using the method of proof of (4) of [2, Theorem 2] one can show that if λ satisfies $|\lambda-N| < N$, then λ is a simple proper value of Γ_a^{1*}. The fact that the spectrum is closed completes the theorem.

We shall now show that H_ε is as required. Indeed,

$$\varepsilon_n = \frac{1}{c}\left[1+\frac{a/c}{n+a-a/c}\right].$$

Thus

$$\|H_\varepsilon\|_p \leqq \frac{1}{|c|}+\frac{a}{|c|^2}\,\|H_\delta\|_p,$$

where $\delta_n = \dfrac{1}{n+a-a/c}$, and the theorem reduces to showing that $\|H_\delta\|_p$ is finite.

Let $x \in l^p$. Then

$$\|H_\delta x\|_p = \left\{\sum_{n=0}^{\infty}\left|\sum_{k=0}^{n} h_{nk}x_k\right|^p\right\}^{1/p} \leqq \left\{\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} |h_{nk}|\,|x_k|\right]^p\right\}^{1/p}.$$

Now

$$|h_{nk}| = \binom{n}{k}|\Delta^{n-k}\mu_k| = \binom{n}{k}\left|\int_0^1 t^{k+a-a/c-1}(1-t)^{n-k}\,dt\right| \leqq$$

$$\leqq \binom{n}{k}\int_0^1 t^{k+a-\mathrm{Re}(a/c)-1}(1-t)^{n-k}\,dt.$$

Thus $\|H_\delta x\|_p \leqq H_{|\delta|}(p)\|x\|_p$, where

$$H_{|\delta|}(p) = \int_0^1 t^{-\frac{1}{p}+a-\mathrm{Re}(a/c)-1}\,dt,$$

provided the integral exists. It remains to show that $a-\mathrm{Re}(a/c)-1/p > 0$.

Note that $c = N-\lambda$. Thus $a-\mathrm{Re}(a/c) = a+a\left[\frac{\alpha-N}{(\alpha-N)^2+\beta^2}\right]$.

By hypothesis $|\lambda|>N$. If we let $\lambda = \alpha+i\beta$, then $|\lambda|>N$ is equivalent to $\alpha^2+\beta^2 > N^2$, which can be written in the form $(\alpha-N)^2+\beta^2 > 2N(N-\alpha)$. Hence $\frac{\alpha-N}{(\alpha-N)^2+\beta^2} > -\frac{1}{2N}$.

The proof is now complete, since $1-1/2N = 1/ap$.

2. Finite continuous methods

Theorem 5. *Let T be an integral Hausdorff transformation defined by*

$$(1) \qquad T(f)(y) = \int_0^1 f(xy)\,dq(x),$$

where q is an absolutely continuous totally regular mass function. Then for each T which is a bounded operator over $L^p[0, 1]$, $\|T\|_p = H(p)$.

Proof. From [5, p. 243], $\|Tf\|_p \leqq H(p)\|f\|_p$, with equality holding only for $f\equiv 0$ or $T\equiv I$.

To prove the reverse inequality, let $f_1(x)=x^{-\beta}$, $\beta > 1/p$. Then $f_1\in L^p[0, 1]$ and $(Tf_1)(y)=y^{-\beta}H(p)$. Therefore $\|T\|_p \geqq H(p)$.

Theorem 6. *For each $T\neq I$, $NI-T$ has a total set of proper vectors corresponding to proper values of modules strictly less than N.*

With $f_n(x)=x^n$, $n=0, 1, 2, \dots$, the family $\{f_0, f_1, f_2, \dots\}$ is total over $L^p[0, 1]$, and $(Tf_n)(y) = \int_0^1 (xy)^n dq(x) = \mu_n y^n$. Therefore $((N-T)f_n)(y) = (N-\mu_n)y^n$.

Since T^* is playing the role over $L^q[0, 1]$ that was played by H over l^p, one conjectures that T^* has empty point spectrum. The conjecture remains to be verfied general, but is true in the following special cases.

Theorem 7. *For α a positive integer, the point spectrum of H_α^* is empty.*

Proof. H_α^* has kernel

$$k^*(x, y) = \begin{cases} 0, & 0<y\leqq x \\ \dfrac{1}{y\Gamma(\alpha)}(\log(y/x))^{\alpha-1}, & 0<x<y. \end{cases}$$

Suppose $H_\alpha^* g = \lambda g$ for some nonzero $g \in L^q[0, 1]$, where q is the conjugate index of p. Then we have

$$\frac{1}{\Gamma(\alpha)} \int_x^1 \frac{1}{y}\left(\log\frac{y}{x}\right)^{\alpha-1} g(y)\,dy = \lambda g(x). \tag{2}$$

If $\lambda = 0$, then differentiating the above gives

$$\int_x^1 \frac{g(y)}{y}\left(\log\frac{y}{x}\right)^{\alpha-2} dy \equiv 0.$$

Differentiating $\alpha-1$ more times leads to $-g(x)/x \equiv 0$ or $g\equiv 0$, a contradiction.

With $\lambda \neq 0$, (2) implies the differentiability of g. Differentiation yields

$$\lambda x g'(x) = -\frac{1}{\Gamma(\alpha-1)} \int_x^1 \frac{g(y)}{y}\left(\log\frac{y}{x}\right)^{\alpha-2} dy \tag{3}$$

Now let $w = g'(x)$, and regard g as a function of t, where $t=\log x$. Then $xg'(x) = D_t w$. Differentiating (3) $(\alpha-2)$ more times yields $\lambda D_t^\alpha w + (-1)^{\alpha+1} w = 0$, which has solution

$$g(x) = \sum_{k=1}^{\alpha} A_k x^{a_k},$$

where each a_k is a root of the auxiliary equation $a^\alpha + (-1)^{\alpha+1} = 0$.

From (2) and (3) it is clear that g and each of its first $\alpha-1$ derivatives vanish at $x=1$, giving rise to the system

$$\sum_{k=1}^{\alpha} A_k = 0; \quad \sum_{k=1}^{\alpha} a_k(a_k-1)\dots(a_k-j)A_k = 0 \qquad (j=0, 1, \dots, \alpha-2),$$

which is equivalent to the system

$$\sum_{k=1}^{\alpha} a_k j A_k = 0 \qquad (j=0, 1, \dots, \alpha-1).$$

This latter system has a Vandermonde determinant. Therefore each $A_k = 0$ and $g\equiv 0$.

Theorem 8. *For α a positive integer, the point spectrum of C_α^* is empty.*

The method of proof is similar to that of Theorem 7. The kernel for C_α^* is

$$k^*(x, y) = \begin{cases} 0, & 0<y\leqq x \\ \dfrac{\alpha}{y}\left(1-\dfrac{x}{y}\right)^{\alpha-1}, & 0>x>y. \end{cases}$$

The condition $C_\alpha^* g = \lambda g$ becomes

$$\lambda g(x) = \int_x^1 \frac{\alpha}{y}\left(1-\frac{x}{y}\right)^{\alpha-1} g(y)\,dy.$$

For $\lambda \neq 0$, the corresponding differential equation is

$$\lambda x^\alpha g^{(\alpha)}(x) - (-1)^\alpha \Gamma(\alpha+1) g(x) = 0,$$

which is of Euler-type, with solution

$$g(x) = \sum_{k=1}^{\alpha} A_k x^{a_k}.$$

As before, each A_k is zero so that $g \equiv 0$.

Theorem 9. *Γ_a^{1*} has empty point spectrum.*

The kernel for Γ_a^{1*} is

$$k^*(x, y) = \begin{cases} 0, & 0<y\leqq x \\ \dfrac{a}{y}\left(\dfrac{x}{y}\right)^{a-1}, & 0<x<y. \end{cases}$$

The condition $\Gamma_a^{1*} g = \lambda g$ leads for $\lambda \neq 0$ to the differential equation $\lambda x g'(x) = [\lambda(a-1)-a]g(x)$, which has solution $g(x) = Cx^{a-1-a/\lambda}$. Since $g(1)=0$, $C=0$, and $g\equiv 0$.

Theorem 10. $\sigma(\Gamma_a^1) = \{\lambda : |\lambda - N| \leqq N\}$.

To prove $\sigma(\Gamma_a^1) \subseteqq \{\lambda : |\lambda - N| \leqq N\}$ apply the corresponding argument of Theorem 4 to $L^p[0, 1]$.

For the opposite inclusion, suppose

$$\left(1 - \frac{1}{N}\Gamma_a^1\right) f(x) = \lambda f(x).$$

The resulting differential equation has solution

$$f(x) = c_1 \exp\left[-a\left(1 - 1/N(1-\lambda)\right)\log x\right].$$

It is a straightforward exercise to verify that if $|\lambda| < 1$, then $f \in L^p[0, 1]$. Therefore point spectrum of $\left(1 - \dfrac{1}{N}\Gamma_a^1\right)$ contains the open disc $\{\lambda : |\lambda| < 1\}$. Hence $\sigma(\Gamma_a^1) \supset \{\lambda : |\lambda - N| < N\}$. The proof is now complete since the spectrum is closed.

Specializing to $L^2[0, 1]$ we have the following result. Let be a continuous bounded operator over $L_2[0, 1]$ with kernel

$$k(x, y) = \begin{cases} 0, & 0<x\leqq y \\ \dfrac{1}{x} f\left(\dfrac{y}{x}\right), & 0<y<x \end{cases} \tag{4}$$

and adjoint kernel

$$k^*(x, y) = \begin{cases} 0, & 0 < y \leqq x \\ \dfrac{1}{y} f\left(\dfrac{x}{y}\right), & 0 < x < y, \end{cases}$$

where f is nonnegative and integrable over $[0, 1]$.

Theorem 11. *T^* is hyponormal.*

Proof. The kernels for TT^* and T^*T are

$$I_1 = \int_0^1 k(x, u) k^*(u, y)\, du = \frac{1}{xy} \int_0^{\min(x,y)} f\left(\frac{u}{x}\right) f\left(\frac{u}{y}\right) du$$

and

$$I_2 = \int_0^1 k^*(x, u) k(u, y)\, du = -\frac{1}{xy} \int_{\min(x,p)}^{xy} f(z/y) f(z/x)\, dz.$$

Hence

$$I_1 - I_2 = \frac{1}{xy} \int_0^{xy} f\left(\frac{z}{y}\right) f\left(\frac{z}{x}\right) dz = \int_0^1 f(wx) f(wy)\, dw.$$

For any $g \in L^2[0, 1]$,

$$((I_1 - I_2) g, g) = \int_0^1 \int_0^1 g(y) \int_0^1 f(wx) f(wy)\, dw\, dy\, \bar{g}(x)\, dx = \int_0^1 (Fg, Fg)\, du,$$

where $F(g)(w) = \int_0^1 g(y) f(wy)\, dy$.

If g is absolutely continuous then we may write $g(x) = \int_0^x h(t)\, dt$. An elementary change of variable in (1) will change h to the form in (4). Thus every totally regular integral method with absolutely continuous mass function will have its adjoint hyponormal.

3. Infinite continuous methods

Theorem 12. *Let T be a bounded linear operator over $L^p[0, \infty)$ with kernel defined by* (4). *Then* $\|T\|_p = H(p)$.

This theorem is a special case of [5, Th. 319].

Theorem 13. *For α a positive integer H_α and H_α^* have empty point spectra.*

For a proof, combine the facts that $\sigma(H_\alpha) = (\alpha(H))^\alpha$ and that $H = C_1$ has empty point spectrum. The same applies to H_α^*.

Theorem 14. *For α a positive integer C_α and C_α^* have empty point spectra.*

For each operator the proof is similar to that of Theorem 8. In each case one shows that the only solution function in the appropriate space is the zero function.

Theorem 15. *Γ_a^1 and Γ_a^{1*} have empty point spectra.*

The proofs here parallel that of Theorem 9.

Theorem 16. $\sigma(\Gamma_a^1) = \{\lambda : |\lambda - N| = N\}$.

To prove this theorem one follows the argument of [1] using

$$P(x)(t) = \frac{1}{t}\int_0^t x(s)\,a\left(\frac{s}{t}\right)^{a-1} ds$$

with

$$P_\zeta x(t) = \int_0^1 a x(st) s^{a(1-\zeta)-1}\,ds \quad \text{and} \quad Q_\zeta x(t) = \int_0^\infty a x(s) s^{a(1-\zeta)-1}\,ds$$

as the corresponding resolvents in the appropriate regions.

For $p=2$ we have the following result.

Theorem 17. *Let T be a continuous bounded operator over $L_2[0, 1]$ with kernel as in* (4) *with f now integrable over* $[0, \infty)$. *Then T^* is normal.*

The kernels corresponding to TT^* and T^*T are

$$I_1 = \int_0^\infty k(x, u)k^*(u, y)\,du = \frac{1}{xy}\int_0^{\min(x,y)} f\left(\frac{u}{x}\right) f\left(\frac{u}{y}\right) du$$

and

$$I_2 = \int_0^\infty k^*(x, u)k(u, y)\,du = \frac{1}{xy}\int_0^{\min(x,y)} f\left(\frac{z}{y}\right) f\left(\frac{z}{x}\right) dz;$$

hence $I_1 - I_2 \equiv 0$.

The author takes this opportunity to acknowledge at least one valuable conversation with each of the following: A. Brown, P. Halmos, Donald J. Kerr, Jr., A. Shields, and J. P. Williams.

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(Received December 8, 1969)