The value distribution of composite entire functions

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1. If the entire function $F(z)$ is expressible in the form $f \circ g(z)$, where f and g are transcendental entire functions, it is called composite; otherwise $F(z)$ is said to be pseudo-prime. OZAWA [3] proved various results about the value distribution of composite entire functions, including the following:

If $F(z)$ is entire and of finite order and if there exists a constant A such that $F(z)=A$ has only real roots, then $F(z)$ is not composite.

Thus a composite entire function $F(z)$ of finite order has none of its A-values distributed entirely on a line and, a fortiori none is distributed on a ray. One can strengthen this last statement and assert that there is no direction which is the sole limiting direction of the A-points:

Theorem 1. *If $F(z)$ is an entire function of finite order and there exist complex A and real α such that for any $\delta>0$ all but a finite number of roots of $F(z)=A$ lie in the angle $|\arg z-\alpha|<\delta$, then $F(z)$ is pseudo-prime.*

In Section 3 similar arguments to those used in the proof of Theorem 1 are applied to a question of iteration theory.

2. Proof of Theorem 1. (i) Without loss of generality, we may suppose $\alpha=\pi$. Suppose $F(z)$ satisfies the conditions of the theorem and that, nevertheless, $F=f(g)$, f and g are transcendental. Then by a result of PÓLYA [4], f has zero order and g has finite order (less than that of F).

Now $f(w)=A$ has an infinity of solutions $w=w_1, w_2, \ldots, w_n, \ldots$ and $|w_n|\to\infty$. For any $\delta>0$, the roots of $g(z)=w_n$ $(n>n_0)$ all lie in the angle $A(\delta)$: $|\arg z-\pi|<\delta$ and so $g(z)$ omits the values w_n in $B(\pi-\delta)$: $|\arg z|\leqq\pi-\delta$.

BIEBERBACH [2] has shown that if the entire function $h(z)$ takes two different finite values at most finitely often in an angle of aperture $\alpha\pi$, then in every smaller angle

$$|f(z)|=O\{\exp(K|z|^{1/\alpha})\}$$

for a suitable constant K.

We deduce that $g(z)$ is of order $\leqq \frac{1}{2}\pi/(\pi-\delta)$ in $B(\pi-2\delta)$. Since $g(z)$ is of some finite order, say ϱ, in the whole plane, in particular in $A(2\delta)$ of aperture 4δ, δ arbitrary, it follows from the Phragmén—Lindelöf principle that $\varrho \leqq \frac{1}{2}$.

(ii) Choose $w_k \neq g(0)$ and $0<\delta<\pi/16$. Then $g(z)$ may be expressed

$$g(z)-w_k=\lambda \prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right)=P(z)\prod_{n=n_0}^{\infty}\left(1-\frac{z}{z_n}\right),$$

$0 \neq \lambda$ constant, $P(z)$ polynomial. Since $f(g(z_n))=A$ we may assume that for given $\delta>0$, $z_n \in A(\delta)$ when $n \geqq n_0$.

For $z \in B\left(\frac{\pi}{2}-\delta\right)$: $|\arg z|<\frac{\pi}{2}-\delta$ and for $n \geqq n_0$ we then have $|\arg(-z/z_n)|<\frac{\pi}{2}$, and so $\left|1-\frac{z}{z_n}\right|>1$. Thus as $z \to \infty$ in $B\left(\frac{\pi}{2}-\delta\right)$, $|g(z)| \to \infty$ faster than any power of $|z|$.

(iii) Next we show that for all large enough z, ($|z|>K$), i.e. in $|\arg z|<\delta$, we have for

(1) $$D=zg'(z)/\{g(z)-w_k\}=\sum_{n=1}^{\infty} z/(z-z_n)$$

that

(2) $$|D|>4\pi\delta^{-1}, \quad |\arg D|<2\delta.$$

First note that the bilinear function $t=z/(z-\beta)$ maps the line joining β to $-\beta$ onto the real axis and the angle $|\arg z-\arg(-\beta)|<2\delta$ into the region E bounded by the two circular arcs joining 0 to 1 and making angles $\pm 2\delta$ with the positive real axis at 0. Hence for $n \geqq n_0$, when $z_n \in A(\delta)$, $t=z/(z-z_n)$ maps $B(\delta)$, which belongs to $|\arg z-\arg(-z_n)|<2\delta$, into E, and so for each $n \geqq n_0$

(3) $$|\mathrm{Im}\, z/(z-z_n)| \leqq \tan(2\delta)\cdot \mathrm{Re}\, z/(z-z_k) \quad \text{in} \quad B(\delta).$$

Since, for each fixed n, $z/(z-z_n) \to 1$ as $z \to \infty$, one has for all $z \in B(\delta)$ with sufficiently large $|z|$, that (3) holds for all n. Hence, from (1),

$$|D| \geqq \mathrm{Re}\, D>4\pi\delta^{-1},$$

and

$$|\mathrm{Im}\, D|<(\tan 2\delta)\cdot \mathrm{Re}\, D, \quad |\arg D|<2\delta$$

for $z \in B(\delta)$, $|z|>K$, say.

(iv) Choose w_n, $n>n_0$, such that $f(w_n)=A$ and so large that $|g(z)-w_k|<|w_n-w_k|$ for $|z| \leqq K$, where K is the constant which occurs in (iii). The component C of the set $\{z: |g(z)-w_k|<|w_n-w_k|\}$, which contains the origin, has a bounded intersection with $B(\delta)$ and this intersection contains $B(\delta) \cap \{|z| \leqq K\}$. Then the boundary of $C\cap B(\delta)$ contains an arc of a level curve γ of $g(z)-w_k$ which joins a point of $\arg z=-\delta$ to a point of $\arg z=\delta$ and lies in $|z|>K$. On γ one has (1) and (2) of (iii). Hence the arc γ contains no zeros of $g'(z)$. If, moreover, an increment δz on γ corresponds to an increment δw on $|w-w_k|=|w_n-w_k|$ under $w=g(z)$, then

$$\frac{\delta w}{(w-w_k)}=\frac{\delta z}{z}\cdot\frac{zg'(z)}{g(z)-w_k}\{1+o(\delta z)\}, \tag{4}$$

so that

$$\arg\left(\frac{\delta z}{z}\right)-\frac{\pi}{2}=\arg\left(\frac{\delta z}{z}\cdot\frac{w-w_k}{\delta w}\right)=-\arg\frac{zg'(z)}{(g(z)-w_k)}\{1+o(\delta z)\},$$

and by (2) $\left|\arg\left\{\frac{zg'(z)}{g(z)-w_k}\right\}\right|<2\delta<\frac{\pi}{8}$, so the arc γ can be expressed as $z=r(\theta)e^{i\theta}$, $-\delta\leqq\theta\leqq\delta$.

Putting $w-w_k=|w_n-w_k|e^{i\varphi}$, we have in (4):

$$i\delta\varphi\{1+o(\delta\theta)\}=\left\{\frac{\delta r}{r}+i\cdot\delta\theta\right\}\left\{\frac{zg'(z)}{g(z)-w_k}\right\}\{1+o(\delta\theta)\},$$

whence

$$\left|\frac{\partial\varphi}{\partial\theta}\right|\geqq\left|\frac{zg'(z)}{g(z)-w_k}\right|>4\pi\delta^{-1},\ \text{by (2).}$$

As z traverses γ in the direction of increasing θ, w traverses the circle Γ: $|w-w_k|=|w_n-w_k|$ in the positive direction and φ increases by at least $4\pi\delta^{-1}\cdot 2\delta=8\pi$. Thus w traverses the whole of Γ and in particular $g(z)=w=w_n$ for some point $z\in\gamma\subset B(\delta)$. But this contradicts the fact, established in (i), that $g(z)=w_n, n>n_0$, has no roots outside $A(\delta)$. Thus the assumption that $F(z)$ is composite must be false.

3. A related question in iteration theory. Let $f(z)$ be an entire function and $f_1(z)=f(z)$, $f_2(z)=f(f(z)),\dots,f_n(z),\dots$ be its sequence of iterates. Regarding the Fatou set $\mathfrak{F}(f)$ of those points of the complex plane where $\{f_n(z)\}$ does not form a normal family, it was shown in [1] that *if $f(z)$ is entire and transcendental, then $\mathfrak{F}(f)$ cannot be contained in any finite set of lines* but on the other hand, *for any constant $A>0$ there exists an entire transcendental function for which $\mathfrak{F}(f)$ is contained in the region $\{|\mathrm{Im}\, z|<A, \mathrm{Re}\, z>0\}$.*

The function used to show this last result was of infinite order. In fact, using the arguments of Section 2 we can show:

Theorem 2. *If f is entire transcendental and for every $\delta>0$ the set $\mathfrak{F}(f)-$ $-\{z, (\arg z)<\delta\}$ is bounded, then f is of infinite order.*

Proof. Suppose f satisfies the hypotheses of the theorem, but is of finite order. $\mathfrak{F}(f)$ has the properties (cf. [1]):

(i) $\mathfrak{F}(f)$ is non-empty and perfect,
(ii) If $f(z)=\alpha\in\mathfrak{F}$, then $z\in\mathfrak{F}$.

We take two different values α, β in $\mathfrak{F}(f)$ which are not Picard exceptional for $f(z)$. The solutions of $f(z)=\alpha, \beta$ lie in $\mathfrak{F}$ and so, with finitely many exceptions in $|\arg z|<\delta$. Noting that $\delta>0$ is arbitrary and proceeding as in § 2 (i), we see that $f(z)$ has order at most $\frac{1}{2}$.

The method of Section 2 (ii)—(iv) then shows that in the angle B: $|(\arg z)-\pi| < \delta$ obtained from $|\arg z|<\delta$ by reflection in the origin, $f(z)$ takes all arbitrarily large values, in particular large values z_n for which $f(z_n)=\alpha$, i.e. values for which $z_n\in\mathfrak{F}$. If $f(t_n)=z_n$, $t_n\in B$, we have $t_n\in\mathfrak{F}$, since $z_n\in\mathfrak{F}$. Taking a sequence $z_n\in\mathfrak{F}$ for which $|z_n|\to\infty$, we have $|t_n|\to\infty$ and hence $\mathfrak{F}\cap B$ is unbounded or $\mathfrak{F}-\{z, |\arg z| < \delta\}$ is unbounded, against the assumptions of the theorem. Hence f must be of finite order.

In Theorem 2 the transcendence of f is essential. Polynomials have bounded $\mathfrak{F}$.

References

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