Generalization of a theorem of A. and C. Rényi on periodic functions

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In memory of Alfred and Catherina Rényi

A. and C. RÉNYI proved [4]

Theorem A. *If $f(z)$ is an entire function and $p(z)$ is a polynomial of degree $n \geqq 3$, then $f[p(z)]$ can not be periodic.*

We prove now the following generalization to meromorphic function:

Theorem. *Let $f(z)$ be a non-constant meromorphic function and let $p(z)$ be a polynomial of degree n. The function*

$$F(z) = f[p(z)]$$

can not be periodic unless n has one of the values 1, 2, 3, 4, 6.

If $n=1$, then $F(z)$ can be any periodic, meromorphic function. If $n=2$, then $F(z)$ is obtained by simple changes of variable from an even periodic function. If $n \geqq 3$ then F is an elliptic function and $F(z) = g[(z+\alpha)^n]$ for a suitable meromorphic g and complex α.

Lemma. *Let*

$$p(z) = az^n + bz^{n-\nu} + \cdots \qquad (\nu \geqq 2)$$

be a polynomial of degree n. If $|z|$ is sufficiently large ($|z| > r_0$), then the roots ς of

$$p(\varsigma) = p(z) \qquad (|z| > r_0)$$

are given by

$$\varsigma = \varrho^k z + O\left(\frac{1}{|z|}\right), \qquad (k = 1, 2, \ldots, n),$$

where

$$\varrho = e^{2\pi i/n}.$$

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Proof (Lemma): a simple application of the implicit function theorem to the equation

$$\left(p(\varsigma)\right)^{-\frac{1}{n}} = \varrho^{-k}\left(p(z)\right)^{-\frac{1}{n}}$$

regarding $\frac{1}{z}$ and $\frac{1}{\varsigma}$ as the basic variables.

Proof (Theorem). For $n=1$ there is nothing to prove. For $n=2$ we have, completing the square

$$f\left(p(z)\right) = f(az^2+bz+c) = f\left[a\left(z+\frac{b}{2a}\right)^2+c-(b^2/4a)\right]$$

and $F(z)$ is an even function of $z+b/2a$.

Suppose now that $n>2$ and that F is periodic. By a simple shift of origin in the z-plane we may assume

$$p(z) = az^n+bz^{n-\nu}+\cdots \qquad (\nu \geqq 2).$$

By replacing z by γz we may also suppose $F(z) = F(z+1)$. Choose z quite arbitrarily For a sufficiently large integer m the equation

$$p(\varsigma) = p(z+m)$$

has a solution

(1) $$\varsigma = \varrho(z+m)+o(1) \qquad (m\to\infty).$$

Also, if m is sufficiently large, $|\varsigma+m'|$ will be greater than r_0 (of the Lemma) for every integer m'.

From the properties of F

(2) $$F(\varsigma)=F(z).$$

Again, with ς as just defined

$$p(\varsigma') = p(\varsigma+m')$$

has a root

$$\varsigma' = \varrho(\varsigma+m')+o(1) = \varrho^2 z+\varrho^2 m+\varrho m'+o(1). \qquad (m\to\infty)$$

Also

$$F(\varsigma'+m) = F(\varsigma') = F(\varsigma) = F(z).$$

i.e. for given z the equation

(3) $$F(w)=F(z)$$

has solutions

(4) $$w = \varrho^2 z+\varrho(\varrho m+m'+\varrho^{-1}m)+o(1) \qquad (|m|>M_0,\ m' \text{ arbitrary}).$$

Now $\varrho m + \varrho^{-1} m + m' = \left(2\cos\frac{2\pi}{n}\right) m + m'$.

If $2\cos\frac{2\pi}{n}$ is irrational then $\left(2\cos\frac{2\pi}{n}\right) m + m'$ can be made arbitrarily close to any real number ξ for some arbitrarily large integer m and corresponding suitable m'. This means

$$F(\varrho^2 z + \varrho\xi) \equiv F(z) \qquad (-\infty < \xi < \infty)$$

and so F must be a constant, and the same is true of f.

If $\alpha = \cos\frac{2\pi}{n}$ is rational, then the primitive n^{th} root of unity ϱ satisfies $\varrho^2 - 2\alpha\varrho + 1 = 0$.

But the primitive n^{th} roots of unity obey an irreducible equation of degree $\varphi(n)$, $g(\varrho) = 0$; $g(\varrho)$ must divide $\varrho^2 - 2\alpha\varrho + 1$, so that $\varphi(n) = 1$ or $\varphi(n) = 2$.

We have

$$\varphi(n) = n \prod_{p/n} \left(1 - \frac{1}{p}\right) \geqq \prod (p-1).$$

If $\varphi(n) \leqq 2$, the only possible prime factors of n are 2 and 3 and it is now immediate that $n = 3$, 4 or 6.

If $2\cos\frac{2\pi}{n}$ is rational, we can find arbitrarily large m and corresponding m' so that

$$2\cos\frac{2\pi}{n} m + m' = 0.$$

Choosing m and m' in this way and letting $m \to \infty$ we find from (3) and (4)

$$F(\varrho^2 z) = F(z).$$

In the same way, making

$$2\cos\frac{2\pi}{n} m + m' = 1, \quad F(\varrho^2 z + \varrho) = F(z) = F(\varrho^2 z).$$

Therefore F has period ϱ and F is a meromorphic function with the periods 1 and ϱ, i.e., an elliptic function. Also, by (1) and (2)

$$F(\varrho z + \varrho m + o(1)) = F(\varrho z + o(1)) = F(z).$$

In the limit $m \to \infty$

$$F(\varrho z) = F(z).$$

This shows that F is a function of z^n only and the Theorem is proved.

This result proves a conjecture in [1] and resolves problems raised in [2] and [3].

Bibliography

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