Elementary estimates for certain types of integers

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1. Introduction. For each integer $k \geqq 2$, let L_k represent the set of positive integers n such that each prime factor of n occurs with multiplicity at least k. Let $l_k(n)$ denote the characteristic function of the set L_k, and for real $x \geqq 1$, let $L_k(x)$ be the number of integers contained in L_k and not exceeding x. Let Q be the set of squarefree integers and $q(n)$ the characteristic function of Q. The Riemann zeta-function will be denoted $\zeta(s)$ for real s.

The starred references of this paper refer to the bibliography of the paper [2] by the first author. All O-constants which occur are understood to depend upon k.

In 1934 ERDŐS and SZEKERES [5*] obtained the following estimate for $L_k(x)$:

$$L_k(x) = c_k x^{1/k} + O(x^{1/(k+1)}) \tag{1.1}$$

where c_k is a constant. This was proved by elementary means without any essential use of Dirichlet series. Later BATEMAN and GROSSWALD obtained (1. 1) in the stronger form

$$L_k(x) = c_k x^{1/k} + c_k' x^{1/(k+1)} + O(x^{1/(2k+1)}), \tag{1.2}$$

where c_k', like c_k, is independent of x. While the Bateman—Grosswald proof is elementary, it makes use of the uniqueness theorem for Dirichlet series (see Remark 1 below).

It is the purpose of the present paper to establish certain weaker estimates for $L_k(x)$ by strictly elementary methods. In particular, we show in § 6, without appealing to the uniqueness theorem, that

$$L_k(x) = c_k x^{1/k} + c_k' x^{1/(k+1)} + O(x^{1/(k+2)}). \tag{1.3}$$

The argument used in the paper is an elaboration of the method of ERDŐS and SZEKERES [5*]. We require, in addition, estimates for some special sums (§ 4) and an asymptotic formula for the average of a certain divisor function (§ 5). In § 7 we give a simple, independent proof of the slightly weaker form of (1. 3) with the O-term $O(x^{1/(k+2)}\log x)$.

Remark 1. The case $k=2$ is exceptional with respect to the above discussion of (1. 2). In fact, an elementary proof of (1. 2) in this case has been given by BATEMAN [1*]; also see [2] and [3, § 3].

2. Density of L_k. Our first estimate for $L_k(x)$ is given in the following theorem. Let $L_2=L$.

Theorem 1. *The set L has density* 0; *that is,*

$$\lim_{x\to\infty} \frac{L(x)}{x} = 0.$$

Proofs of this result have been given by FELLER and TOURNIER [6*, § 9] and SCHOENBERG [10*, § 12]. The corresponding result for L_k, $k\geqq 2$ follows immediately.

3. O-estimate for $L_k(x)$. We first prove a characterization of the set L_k.

Lemma 1. *A necessary and sufficient condition that an integer n be in L_k is that it admit a representation of the form*

$$n = d_1 d_2^2 \dots d_{k-1}^{k-1} d^k, \quad d_1 d_2 \dots d_{k-1} | d. \tag{3. 1}$$

Proof. Suppose n can be written in the form (3. 1), and let $p|n$, p prime. Then $p|d$ and hence $p^k|n$. This proves the sufficiency.

Now suppose $n\in L_k$, $n=p_1^{e_1}\dots p_s^{e_s}$, $e_i\geqq k$ $(i=1, \dots, s)$ where $p_1, \dots, p_s$ are the distinct prime divisors of n. Now $e_i = q_i k + r_i$, $q_i>0$, $0\leqq r_i<k$ $(i=1, \dots, s)$. Therefore $p_i^{e_i}=(p_i^{q_i})^k p_i^{r_i}$ for each i, from which it follows that n is expressible in the form (3. 1) in such a way that $d=p_1^{q_1}\dots p_s^{q_s}$ and $d_1 \dots d_{k-1}$ is the product of those p_i for which the corresponding $r_i>0$.

We are now in a position to prove the following result. Throughout this paper the symbol Σ' will indicate that the sum is taken over integers in L_k. Let $[x]$ denote the largest integer $\leqq x$.

Theorem 2. *For* $x\geqq 1$,

$$L_k(x) = O(x^{1/k}) \quad as \quad x\to\infty. \tag{3. 2}$$

Proof. Let $\delta = d_1 d_2^2 \dots d_{k-1}^{k-1}$. By Lemma 1,

$$L_k(x) = \sum_{n\leqq x}{}' 1 \leqq \sum_{\delta d^k \leqq x}{}' 1$$

where the last summation is over all k-tuples of natural numbers $d_1, d_2, \dots, d_{k-1}$, d such that $D = d_1 d_2 d_3 \dots d_{k-1}$ divides d, $DN=d$.

Thus

$$L_k(x) = \sum_{\delta\leqq x} \sum_{d^k \leqq x/\delta} 1.$$

Summing over N, we see that the interior sum has the value,

$$[x^{1/k}/d_1^{1+1/k}d_2^{1+2/k}\dots d_{k-1}^{1+(k-1)/k}].$$

Hence

$$L_k(x) \leqq x^{1/k}\sum_{\delta\leqq x}(d_1^{1+1/k}d_2^{1+2/k}\dots d_{k-1}^{1+(k-1)/k})^{-1} = 0(x^{1/k}),$$

and the theorem is proved.

A different proof of (3. 2) is indicated by HORNFECK in [8*, Lemma 2].

4. Lemmas. This section contains two lemmas which will be needed in the last two sections.

Lemma 2. (a) *For* $0<s<1/k$,

$$\sum_{n\leqq x}{}' \frac{1}{n^s} = O\left(x^{\frac{1}{k}-s}\right), \quad x\geqq 1. \tag{4. 1}$$

(b) *For* $s=1/k$,

$$\sum_{n\leqq x}{}' \frac{1}{n^s} = O(\log x), \quad x\geqq 2. \tag{4. 2}$$

(c) *For* $s>1/k$,

$$\sum_{n>x}{}' \frac{1}{n^s} = O\left(x^{\frac{1}{k}-s}\right), \quad x\geqq 1. \tag{4. 3}$$

Proof. By partial summation and the definition of $l_k(n)$,

$$\sum_{n\leqq x}{}' n^{-s} = \sum_{n\leqq x}\frac{l_k(n)}{n^s} = \sum_{n\leqq x}L_k(n)\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)+\frac{L_k(x)}{([x]+1)^s};$$

hence, by Theorem 2, since $(1+1/n)^s = 1+O(1/n)$,

$$\sum_{n\leqq x}{}' n^{-s} = O\big(\sum_{n\leqq x} n^{-s-(k-1)/k}\big)+O(x^{1/k-s}).$$

If $s\leqq 1/k$, the first O-term in the last expression is $O(x^{1/k-s})$ or $O(\log x)$ according as $s<1/k$ or $s=1/k$. This proves (a) and (b).

Similarly, for $ks>1$, we have with $y>x$,

$$\sum_{y\geqq n>x}{}' n^{-s} = \sum_{y\geqq n>x} L_k(n)\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)-\frac{L_k(x)}{([x]+1)^s}+\frac{L_k(y)}{([y]+1)^s} =$$

$$= O\big(\sum_{n>x} n^{-s-(k-1)/k}\big)+O\left(\frac{1}{x^{s-1/k}}\right),$$

and since both O-terms are $O(x^{1/k-s})$ the lemma results as $y\to\infty$.

We define $\sigma^*(s, n)$ to be the sum of the s-th powers of the square free divisors of n, and $\sigma(s, n)$ to be the sum of the s-th powers of all divisors of n. Place $\theta(n)=\sigma^*(0, n)$.

Lemma 3. *If* $0<\alpha<(k-1)/(k+2)$, *then*

$$\sum_{\substack{n\le x\\ n\in L_{k+2}}} \frac{\sigma^*(-\alpha, n)}{n^{(1+\alpha)/(2k+1)}} = O(x^{1/(k+2)-(1+\alpha)/(2k+1)}). \tag{4. 4}$$

Proof. Place

$$S^*(\alpha, x) = \sum_{\substack{n\le x\\ n\in L_{k+2}}} \frac{\sigma^*(-\alpha, n)}{n^{(1+\alpha)/t}},$$

$$S(\alpha, x) = \sum_{n\le x^{1/e}} \frac{\sigma(-\alpha, n^e)}{n^{e(1+\alpha)/t}},$$

where $e = k+2$, $t = 2k+1$. We estimate $S(\alpha, x)$ first and then reduce the estimation of $S^*(\alpha,x)$ to that of $S(\alpha, x)$. It is convenient to use $\ll$ in place of the O-symbol below.

Noting that $\sigma^*(-\alpha, n^e) = \sigma^*(-\alpha, n)\le\sigma(-\alpha, n)$, one obtains

$$S(\alpha, x) \le \sum_{n\le x^{1/e}} \frac{\sigma(-\alpha, n)}{n^{e(\alpha+1)/t}} =$$

$$= \sum_{n\le x^{1/e}} \sum_{d\delta=n} d^{-\alpha}(d\delta)^{-e(\alpha+1)/t} = \sum_{d\le x^{1/e}} d^{-\alpha-e(\alpha+1)/t} \sum_{\delta\le x^{1/e}/d} \delta^{-e(\alpha+1)/t}.$$

Since $(k+2)(\alpha+1)<2k+1$, it follows that

$$S(\alpha, x) \ll \sum_{d\le x^{1/e}} d^{-\alpha-e(\alpha+1)/t} \left(\frac{x^{1/e}}{d}\right)^{1-e(\alpha+1)/t} \ll x^{(1/e-(\alpha+1)/t)} \sum_{d\le x^{1/e}} d^{-\alpha-1} \ll$$

$$\ll x^{(1/e-(\alpha+1)/t)},$$

in view of the positivity of α.

We observe that every integer n of L_{k+2} has a unique representation of the form $n=pm^{k+2}$, where $p=p_1^{e_1}p_2^{e_2}\dots p^{e^r}, p_1, p_2, \dots, p_r$ being distinct primes, $p_1<p_2<\dots<p_r$, and $e_i>k+2$, $(k+2)$ does not divide e_i, p_i does not divide m, for each i. Therefore,

$$S^*(\alpha, x) = \sum_{r\ge 0} \sum \sum_{\substack{m^e p\le x\\ p_i\nmid m}} \frac{\sigma^*(-\alpha, pm^e)}{p^{(\alpha+1)/t} m^{e(\alpha+1)/t}}$$

where the second summation is over all ordered r-tuples of natural numbers $e_1, e_2, \dots, e_r$ such that e does not divide $e_i, e_i>e$ $(i=1, \dots, r)$ and all r-tuples of prime numbers $p_1, p_2, \dots, p_r$ such that $p_1<p_2<\dots<p_r$ (being vacuous in case $r=0$). By the multiplicative nature of σ^*, $\sigma^*(-\alpha, m^e p) = \sigma^*(-\alpha, m^e)\sigma^*(-\alpha, p)$. After applying this property we drop the condition that p_i does not divide m in the third summation, getting

$$S^*(\alpha, x) \le \sum_{r\ge 0} \sum \sigma^*(-\alpha, p)\, S(\alpha, x/p)/p^{(\alpha+1)/t}$$

where the second summation is over the same natural numbers as before. By the above estimate for $S(\alpha, x)$, we get, dropping the condition that e does not divide e_i but retaining the other conditions, and placing

$$\beta = \frac{1}{e} - (\alpha+1)/t,$$

$$S^*(\alpha, x) \ll x^{\beta} \cdot \sum_{r=0}^{\infty} \sum \frac{\sigma^*(-\alpha, p)}{p^{1/e}} = x^{\beta} \sum_{\substack{n=0 \\ n \in L_{e+1}}}^{\infty} \frac{\sigma^*(-\alpha, n)}{n^{1/e}} \leqq x^{\beta} \sum_{\substack{n=0 \\ n \in L_{e+1}}}^{\infty} \frac{\theta(n)}{n^{1/e}}.$$

It follows that it suffices to show the convergence of the series on the right. A formal computation gives

$$\sum_{\substack{n=0 \\ n \in L_{e+1}}}^{\infty} \frac{\theta(n)}{n^{1/e}} = \prod_{p} \left\{1 + \left(\frac{2}{p^{1+1/e}}\right)\left(\frac{1}{1-p^{-1/e}}\right)\right\}.$$

Observing that $(1-p^{-1/e})^{-1} \leqq (1-2^{-1/e})^{-1}$ for all p, it follows that the product, and hence the series, converges.

5. A divisor function. We first recall a known estimate for the Legendre totient function $\varphi(x, n)$, which denotes the number of positive integers $\leqq x$ prime to n.

Lemma 4 (cf. [1]). *If* $0 \leqq \alpha < 1$, *then*

(5. 1) $$\varphi(x, n) = \varphi(n)x/n + 0\ (x^{\alpha}\sigma^*(-\alpha, n)),$$

where $\varphi(n) = \varphi(n, n)$, *uniformly in both* x *and* n.

The case $\alpha = 0$ of the following lemma is Lemma 3. 1 of [4]. The general case is proved similarly except that the 0-term of formule (3. 5) of that paper is replaced by $O(x^{\alpha-s})$ in the proof.

Lemma 5. *If* $s > 0$, $s \neq 1$, $x \geqq 1$, *then for* $1 > \alpha \geqq 0$,

(5. 2) $$N_s(x, r) = \sum_{\substack{n \leqq x \\ (n, r)=1}} 1/n^s = \zeta(s)\varphi_s(r)/r^s - \varphi(r)/r(s-1)x^{s-1} + O(x^{\alpha-s}\sigma^*(-\alpha, r)),$$

uniformly in x *and* r, *where* $\varphi_s(r) = \sum_{d|r} \mu(d)(r/d)^s$, $\varphi_1(r) = \varphi(r)$, μ *denoting the Mobius function.*

Now suppose a, b, h and m to be positive integers. For positive integers n, let $\tau_{a,b}^{m,k}(n)$ denote the number of decompositions of n in the form $n = d^a f^b$ where $(d, m) = (f, h) = 1$. We now are ready to prove the main result of this section, an estimate for the summatory function $T_{a,b}^{m,h}(x)$ of $\tau_{a,b}^{m,h}(n)$.

Put $c=a+b$, $r=a/b$, $s=b/a$.

Theorem 3. (cf. 4, Theorem 3. 1 in case $m=h$) *If* $b>a\geqq 1$, $r>\alpha\geqq 0$, *then for* $x\geqq 1$

$$T_{a,b}^{m,h}(x)=a_{m,h}x^{1/a}+b_{m,h}x^{1/b}+O\big(x^{(\alpha+1)/c}\varrho_\alpha(h,m)\big),$$

where $\varrho_\alpha(h,m)=max\,(\sigma^*(-\alpha,h),\sigma^*(-\alpha,m))$,

$$a_{m,h}=\zeta(s)\varphi(m)\varphi_s(h)/mh^s,\quad b_{m,h}=\zeta(r)\varphi(h)\varphi_r(m)/hm^r.$$

Proof. We have

$$T_{a,b}^{m,h}(x)=\sum_{n\leqq x}\tau_{a,b}^{m,h}(n)=\sum_{d^af^b\leqq x}1$$

where in the last sum, $(d,m)=(f,h)=1$.

Thus

$$T_{a,b}^{m,h}(x)=\sum_{d\leqq x^{1/c}}1+\sum_{f\leqq x^{1/c}}1-\sum_{d,f\leqq x^{1/c}}1. \tag{5. 3}$$

Since d and f in the summation cannot both simultaneously be $>x^{1/c}$. Each sum of course still has the conditions $d^af^b\leqq x$, $(d,m)=(f,h)=1$. Let these three sums be denoted by $\Sigma_1,\Sigma_2,\Sigma_3$, respectively.

For the first summation one obtains by Lemma 6, since $a\alpha/b<a^2/b^2<1$,

$$\sum\nolimits_1=\sum_{\substack{d\leqq x^{1/c}\\(d,m)=1}}\varphi\left(\frac{x^{1/b}}{d^r},h\right)=x^{1/b}\varphi(h)N_r(x^{1/c},m)/h+Ox^{(\alpha+1)/c}\sigma^*(-\alpha,h).$$

Application of Lemma 7, gives

$$\sum\nolimits_1=x^{1/b}\zeta(r)\varphi(h)\varphi_r(m)/hm^r-\frac{b}{a-b}\frac{\varphi(h)}{h}\frac{\varphi(m)}{m}x^{2/c}+O\big(\varrho_\alpha(h,m)x^{(\alpha+1)/c}\big), \tag{5. 4}$$

and on applying a similar argument to Σ_2 and Σ_3,

$$\sum\nolimits_2=\zeta(s)\frac{\varphi(m)}{m}\frac{\varphi_s(h)}{h^s}x^{1/a}+\frac{a}{a-b}\frac{\varphi(m)}{m}\frac{\varphi(h)}{h}x^{2/c}+O\big(\varrho_\alpha(h,m)x^{(\alpha+1)/c}\big), \tag{5. 5}$$

$$\sum\nolimits_3=\frac{\varphi(h)}{h}\frac{\varphi(m)}{m}x^{2/c}+O\big(x^{(\alpha+1)/c}\varrho_\alpha(h,m)\big). \tag{5. 6}$$

The theorem results on the basis of (5. 3), (5. 4), (5. 5), and (5. 6).

6. Asymptotic estimation of $L_k(x)$. We first introduce some notation and point out a few elementary facts that will be useful for the later discussion. We denote by A_k, B_k the sets of those positive integers all of whose prime divisors have multiplicity on the ranges $k+1\leqq t<2k$, and $k+2\leqq t<2k$, respectively, with $B_2=\{1\}$. Note that $B_k\subseteqq L_{k+2}$.

Remark 2. If $a \in L_k$, then a has a unique factorization $a = d^k e$ where $e \in A_k$.

Remark 3. If $e \in A_k$, then e has a unique factorization $e = g^{k+1} h$, where $g \in Q$, $h \in B_k$ and $(g, h) = 1$.

The following result is well known.

Remark 4. For positive integers n, $q(n) = \sum_{d^2 e = n} \mu(e)$.

The proof of our main result depends upon the following representation of $l_k(n)$.

Lemma 8.

$$l_k(n) = \sum_{d^k e^{2k+2} f^{k+1} h = n} \mu(e),$$

where the summation is over integers d, e, f and h such that $h \in B_k$ and $(e, h) = (f, h) = 1$.

Proof. By Remarks 2 and 3

$$l_k(n) = \sum_{\substack{d^k e = n \\ e \in A_k}} 1 = \sum_{\substack{d^k g^{k+1} h = n \\ g \in Q}} 1 = \sum_{d^k g^{k+1} h = n} q(g),$$

the last two sums with the conditions $h \in B_k$, $(g, h) = 1$. The lemma results by Remark 4.

The folowing expansion will be needed (Cf. [4], (3. 4)):

(6. 1) $$\sum_{\substack{n=1 \\ (n,r)=1}} \mu(n)/n^s = r^s/\zeta(s)\varphi_s(r) \qquad (s > 1).$$

Since $\varphi_s(n) = n^s \prod_{p|n} (1 - 1/p^s)$ we have

Lemma 9. *If $s \geqq 1$, then $\varphi_s(n)/n^s$ is bounded; for $s > 1$, $\varphi_s(n)/n^s$ is bounded away from zero. In particular, for each $s > 1$, $\varphi_s(n)$ has the order of magnitude of n^s as $n \to \infty$.*

Put $r = k+1$, $t = 2k+1$.

Theorem 4. *If $x \geqq 2$, then*

(6. 2) $$L_k(x) = c_k x^{1/k} + c_k' x^{1/r} + O(x^{1/(k+2)}),$$

where c_k and c_k' are defined by

$$c_k = \zeta^{-1}(2r/k) \sum_{h=1}^{\infty} a_h \left(\frac{h^{t/k}}{\varphi_{2r/k}(h)} \right), \qquad c_k' = \zeta^{-1}(2) \sum_{h=1}^{\infty} b_h \left(\frac{h^{t/r}}{\varphi_2(h)} \right) \qquad (h \in B_k),$$

and $a_h = a_{1,h}$, $b_h = b_{1,h}$ are defined as in Theorem 3.

Remark 5. Note that a_h and b_h are bounded.

Proof. By Lemma 8.

(6. 3) $$L_k(x) = \sum_{h\leqq x} \sum_{e^{2r}d^k f^r \leqq x/h} \mu(e)$$

with $h \in B_k$ in the first sum and $(e, h)=(f, h)=1$ in the second sum. Let the inner sum of (6. 3) be denoted by Σ^*, $h \leqq x$. Then

$$\Sigma^* = \sum_{\substack{e\leqq(x/h)^{1/2r}\\(e,h)=1}} \mu(e) \quad T_{k,r}^{1,h}(x/he^{2r}),$$

from which, by (6. 1) and by Theorem 3, $(m=1,\ a=k,\ b=k+1)$ with $k/r>\alpha\geqq 0$,

$$\Sigma^* = a_h(x/h)^{1/k} \sum_{\substack{e\leqq(x/h)^{1/2r}\\(e,h)=1}} \frac{\mu(e)}{e^{2r/k}} +$$

$$+ b_h(x, h)^{1/r} \sum_{e\leqq(x/h)^{1/2r}} \mu(e)/e^2 + O\big((x/h)^{(\alpha+1)/t}\sigma^*(-\alpha, h)\big) =$$

$$= x^{1/k} a_h \zeta^{-1}(2r/k) \frac{h^{t/k}}{\varphi_{2r/k}(h)} + b_h x^{1/r}\zeta^{-1}(2)\left(\frac{h^{t/r}}{\varphi_2(h)}\right) + O\big(\sigma^*(-\alpha, h)(x/h)^{(\alpha+1)/t}\big).$$

Substituting this into (6. 3) one deduces by Lemma 9, Remark 5, and the fact that $\beta_k \subseteqq L_{k+2}$,

$$L_k(x) = c_k x^{1/k} + O\big(x^{1/k} \sum_{\substack{h>x\\h\in L_{k+2}}} h^{-1/k}\big) + c_k' x^{1/r} +$$

$$+ O\big(x^{1/r} \sum_{\substack{h>x\\h\in L_{k+2}}} h^{-1/r}\big) + O\left(x^{(1+\alpha)/t} \sum_{\substack{h\leqq x\\h\in L_{k+2}}} \frac{\sigma^*(-\alpha, h)}{h^{(1+\alpha)/t}}\right).$$

By Lemma 4c (with k replaced by $k+2$) the first two O-terms are $O(x^{1/(k+2)})$ and by Lemma 5 (restricting α further to $0<\alpha<(k-1)/(k+2)$) the last is also $O(x^{1/(k+2)})$. This proves Theorem 4.

7. A weaker form of the main result. The argument used to prove Lemma 5 yields the following result for the case $\alpha=0$ of the sum in (4. 4):

(7. 1) $$\sum_{\substack{n\leqq x\\n\in L_{k+2}}} \frac{\theta(n)}{n^{1/(2k+1)}} = O(x^{1/(k+2)-1/(2k+1)}\log x), \qquad x \geqq 2.$$

This result and case $\alpha=0$ in Theorem 3 yield, on the basis of the argument in the preceeding section, the following slightly weaker asymptotic evaluation of $L_k(x)$:

(7. 2) $$L_k(x) = c_k x^{1/k} + c_k' x^{1/(k+1)} + O(x^{1/(k+2)}\log x).$$

This is of interest, in the first place because only the regular form $(\alpha=0)$ of Lemmas 6 and 7 are needed for the proof, and in the second place because (7. 1) can be proved independently in a much simpler way than the corresponding result in Lemma 5.

To prove (7. 1) we recall that $\theta(n)$ denotes the number of square-free divisors of n. By the fundamental theorem of arithmetic, there is a one-to-one correspondence between the square-free divisors of n and the so-called unitary divisors of n (the divisors d of n such that $(d, n/d)=1$); hence $\theta(n)$ is the number of unitary divisors of n. With $t=2k+1$ and $e=k+2$, we have

$$\sum_{\substack{n\leqq x\\ n\in L_e}} \frac{\theta(n)}{n^{1/t}} = \sum_{\substack{n\leqq x\\ n\in L_e}} \frac{1}{n^{1/t}} \sum_{\substack{d\delta=n\\ (d,\delta)=1}} 1 = \sum_{\substack{d\delta\leqq x\\ d\delta\in L_e\\ (d,\delta)=1}} \frac{1}{(d\delta)^{1/t}} = \sum_{\substack{d\leqq x\\ d\in L_e}} \frac{1}{d^{1/t}} \sum_{\substack{\delta\leqq x/d\\ d\in L_e\\ (d,\delta)=1}} \frac{1}{\delta^{1/t}},$$

by the fundamental theorem of arithmetic. We may drop the condition $(d, \delta)=1$ provided the last equality is replaced by inequality ($\leqq$). Lemma 4(a) is then applicable (with k replaced by $k+2$) and its application gives

$$\sum_{\substack{n\leqq x\\ n\in L_e}} \frac{\theta(n)}{n^{1/t}} \ll \sum_{\substack{d\leqq x\\ d\in L_e}} \frac{1}{d^{1/t}} \left(\frac{x}{d}\right)^{1/e-1/t} \ll x^{1/e-1/t} \sum_{\substack{d\leqq x\\ d\in L_e}} \frac{1}{d^{1/e}},$$

and (7. 1) results on applying Lemma 4(b).

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