On the uniform convergence of Fourier transforms on groups

By CONSTANTINE GEORGAKIS in Chicago (Illinois, U.S.A.)

Let G be a non-discrete LCA (locally compact abelian) group with Haar measure dt. Let $f\in L^p(G)$, $1<p<\infty$, $g\in L^q(G)$, $p+q=pq$, and write (t, γ) for the image of an element t in G under the character γ in the dual group Γ of G and f_x for the translate of the function f. For each element x in G, the function $\Phi(x, t)=f_x(t)g(t)$ belongs to $L^1(G)$ and, by the Riemann-Lebesgue theorem, its Fourier transform $\hat{\Phi}(x, \gamma)=\int_G f_x(t)g(t)(-t, \gamma)dt$ converges to zero at infinity on Γ. In this article we show, in particular, the convergence to be uniform with respect to all x in G; that is true also for $p=1$ if G is compact. This is a special case of the more general result Theorem 1, which was first proved by A. PLESSNER for $G=T$, T being the group of real numbers modulo 2π ([2]). In a sense, Corollary 1 and 2 provide an abstract analogue for LCA groups of the classical Riemann—Lebesgue localization principle for Fourier series (see [5]). We need the following theorem of R. R. GOLDBERG and A. B. SIMON ([1], p. 39).

Theorem A. *Let G be a* LCA *group with character group Γ. For each neighborhood V of the identity of G, there exists a compact set K in Γ such that, if $\gamma\notin K$, then* $\operatorname{Re}(t, \gamma)\leqq 0$ *for some $t\in V$.*

Theorem 1. *Let X be an arbitrary set and G a* LCA *group with dual Γ. Let $\Phi(x, t)$ be a complex function on $X\times G$ such that $\int_G|\Phi(x, t)|dt<\infty$ for each $x\in X$; and suppose $\int_G|\Phi(x, t+s)-\Phi(x, t)|dt$ converges to zero as s tends to the identity uniformly for $x\in X$. Then the Fourier transform $\hat{\Phi}(x, \gamma)=\int_G \Phi(x, t)(-t, \gamma)dt$ converges to zero at infinity on Γ uniformly for $x\in X$.*

Proof. Let $\varepsilon>0$. There exists a symmetric neighborhood V of the identity of G so that $\int_G|\Phi(x, t-s)-\Phi(x, t)|dt<\varepsilon$ for all $s\in V$ and $x\in X$. By Theorem A, to this neighborhood V there corresponds a compact set K in Γ so that, if $\gamma\notin K$, there exists an element $s_0\in V$ for which $1-\operatorname{Re}(s_0, \gamma)\geqq 1$. Then we have

$$(s_0, \gamma)\hat{\Phi}(x, \gamma) = \int_G \Phi(x, t)(-t+s_0, \gamma)\, dt = \int_G \Phi(x, t-s_0)(-t, \gamma)\, dt,$$

$$\hat{\Phi}(x, \gamma)\{1-(s_0, \gamma)\} = \int_G \{\Phi(x, t) - \Phi(x, t-s_0)\}(-t, \gamma)\, dt,$$

$$|\hat{\Phi}(x, \gamma)| \leqq |\hat{\Phi}(x, \gamma)|\, |1-\operatorname{Re}(s_0, \gamma)| \leqq |\hat{\Phi}(x, \gamma)\{1-(s_0, \gamma)\}| < \varepsilon$$

for all $\gamma \notin K$ and $x \in X$.

Theorem 2. *Let G be a* LCA *group with dual Γ, $1<p<\infty$, $f \in L^p(G)$, $p+q = pq$ and $g \in L^q(G)$. Then the Fourier transform $\widehat{f_x g}$ converges to zero at infinity on Γ uniformly for $x \in G$.*

Proof. Put $\Phi(x, t) = f_x(t)g(t) = f(x+t)g(t)$ and $X=G$. Then for each s and x in G, we have

$$\int_G |\Phi(x, t+s) - \Phi(x, t)|\, dt = \int_G |f_x(t+s)g(t+s) - f_x(t)g(t)|\, dt \leqq$$

$$\leqq \int_G |f_x(t+s) - f_x(t)|\, |g_s(t)|\, dt + \int_G |f_x(t)|\, |g_s(t) - g(t)|\, dt \leqq$$

$$\leqq \|f_{x+s} - f_x\|_p \|g_s\|_q + \|f_x\|_p \|g_s - g\|_q = \|f_s - f\|_p \|g\|_q + \|f\|_p \|g_s - g\|_q.$$

Since the mapping $s \in G$ to $f_s \in L^p(G)$ is continuous [3], for each $\varepsilon>0$, there exists a neighborhood V of the identity of G so that $\|f_s - f\|_p < \varepsilon/2\|g\|_q$ and $\|g_s - g\|_q < \varepsilon/2\|f\|_p$. Hence,

$$\int_G |\Phi(x, t+s) - \Phi(x, t)|\, dt < \varepsilon \quad \text{for all} \quad s \in V \quad \text{and} \quad x \in G,$$

and the conclusion follows from Theorem 1.

Theorem 3. *Let G be a* LCA *group with dual Γ, $f \in L^1(G)$ and $g \in L^\infty(G)$. $\widehat{f_x g}$ converges to zero at infinity on Γ, if* (a) *or* (b) *or* (c) *below holds*: (a) $\|g_s - g\|_\infty$ *converges to zero at the identity*; (b) $g \in L^p(G)$ *for* $1 \leqq p < \infty$; (c) *for each* $\varepsilon>0$ *there exists a compact subset F in G such that* $\operatorname{ess\,sup}_{t \notin F} |g(t)| < \varepsilon$.

Proof. (a) The case "$f \in L^1(G)$, $g \in L^\infty(G)$ and $\|g_s - g\|_\infty \to 0$ at the identity" can be handled by the argument for Theorem 2.

(b) Let now $f \in L^1(G)$ and $g \in L^\infty(G) \cap L^p(G)$ for $1 \leqq p < \infty$. For $\varepsilon>0$ there exists $h \in C_c(G)$ — the set of continuous functions on G with compact support — such that $\|f-h\|_1 < \varepsilon/\|g\|_\infty$. As in the proof of Theorem 2, we obtain

$$\int_G |\Phi(x, t+s) - \Phi(x, t)|\, dt \leqq \|f_{x+s} - f_x\|_1 \|g\|_\infty + \left|\int_G f_x(t)\{g_s(t) - g(t)\}\, dt\right| \leqq$$

$$\leqq \|f_s - f\|_1 \|g\|_\infty + \left|\int_G h_x(t)\{g_s(t) - g(t)\}\, dt\right| + \left|\int_G \{f_x(t) - h_x(t)\}\{g_s(t) - g(t)\}\, dt\right| \leqq$$

$$\leqq \|f_s - f\|_1 \|g\|_\infty + \|g_s - g\|_p \|h\|_q + 2\|f-h\|_1 \|g\|_\infty < \varepsilon$$

for all $x\in G$ and $s\in V$, where V is an appropriate neighborhood of the identity and $p+q=pq$. Hence, the conclusion follows by Theorem 1.

(c) For a given $\varepsilon>0$ we choose a compact subset F in G so that $\operatorname{ess\,sup}_{t\notin F}|g(t)|\|f\|_1<\varepsilon/2$. Let ξ_1 and ξ_2 be the characteristic functions of F and its complement. According to part (b), there exists a compact subset K_0 in Γ such that $|(\widehat{f_x g\xi_1})(\gamma)|<\varepsilon/2$ for all $x\in G$ and $\gamma\notin K_0$ in Γ. This implies that $|(\widehat{f_x g})(\gamma)|\leqq\varepsilon/2+$ $+\|f_x g\xi_2\|_1\leqq\varepsilon/2+\|f_x\|_1 \operatorname{ess\,sup}_{t\notin F}|g(t)|<\varepsilon$ for $x\notin F$ in G and $\gamma\notin K_0$ in Γ. Since $|(\widehat{f_x g})(\gamma)-(\widehat{f_{x_1} g})(\gamma)|\leqq\|f_x-f_{x_1}\|_1\|g\|_\infty$ and F is compact, there exist elements x_i in F, a neghborhood V of the identity, and compact subsets K_i in Γ $(i=1,2,\ldots,n)$ such that $|(\widehat{f_x g})(\gamma)|<\varepsilon/2$ for $x\in x_i+V$, $\gamma\notin K_i$ in Γ, and $F\subset\bigcup_{i=1}^{n}(x_i+V)$. This shows that $|(\widehat{f_x g})(\gamma)|<\varepsilon$ for all $x\in F$ and $\gamma\notin\bigcup_{i=1}^{n}K_i$ in Γ. Therefore, if $K=\bigcup_{i=0}^{n}K_i$, $|(\widehat{f_x g})(\gamma)|<\varepsilon$ for all $x\in G$ and $\gamma\notin K$ in Γ.

Corollary 1. *If $f\in L^1(G)$ and $g\in C_0(G)$ (the set of all continuous functions on G vanishing at infinity), then* $\lim_{\gamma\in\Gamma}(\widehat{f_x g})(\gamma)=0$ *uniformly for $x\in G$.*

Corollary 2. *If G is compact, $f\in L^p(G)$ and $g\in L^q(G)$, then* $\lim_{\gamma\in\Gamma}(\widehat{f_x g})(\gamma)=0$ *uniformly for $x\in G$.*

Proof. The case $1<p<\infty$ follows from Theorem 2 and the case $p=1$ from Theorem 3.

Remarks. (a) Theorem 2 extends to two parameters as follows: if $f\in L^p(G)$, $g\in L^q(G)$ and $h\in L^r(G)$, where $1<p,q,r<\infty$ and $p^{-1}+q^{-1}+r^{-1}=1$, then the Fourier transform of the function $f_x g_y h$ converges to zero at infinity on Γ uniformly for x and y in G. This follows from Theorem 2 and the inequality

$$\|f_{x+s}g_{y+s}h_s-f_x g_y h\|_1\leqq\|f\|_p\|g\|_q\|h_s-h\|_r+\|f\|_p\|g_s-g\|_q\|h\|_r+\|f_s-f\|_p\|g\|_q\|h\|_r.$$

(b) Theorem 2 remains true also if we replace $L^p(G)$, $L^q(G)$ by a pair of normed homogeneous spaces A, B of functions on G such that $\|fg\|_1\leqq\|f\|_A\|g\|_B$. (A normed space A of functions on G is homogeneous if its norm is translation invariant and the mapping of taking translates is continuous ([4]).)

(c) Our results show that the technique of R. R. Goldberg and A. B. Simon [1] is useful for obtaining refinements of the Riemann—Lebesgue theorem.

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DEPAUL UNIVERSITY
CHICAGO, ILLINOIS
U.S.A.

(Received September 17, 1969)