The Hilbert matrix as a singular integral operator*)

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In this note we show that the theory of the Hilbert matrix $\{1/(j+k+1)\}_{j,k=0}^{\infty}$ can be derived from the theory of selfadjoint singular integral operators, i.e. operators in $L^2(-\infty, \infty)$ of the form

$$(1) \qquad S\colon F(x) \to A(x)F(x) + (\pi i)^{-1}\,\mathrm{P.V.}\int_{-\infty}^{+\infty} (x-t)^{-1}\bar{K}(x)K(t)F(t)\,dt$$

where $A(x), K(x) \in L^\infty\ (-\infty, \infty)$ and $A(x)$ is real valued.

If $f(x) \in L^2(0, \infty)$ its Mellin transform is

$$F(x) = \lim_{a \searrow 0} \int_a^{1/a} t^{-1/2+ix} f(t)\,dt$$

where the limit is taken in the metric of $L^2(-\infty, \infty)$. We have then

$$2\pi f(x) = \lim_{A\to\infty} \int_{-A}^{+A} x^{-1/2-it} F(t)\,dt$$

where the limit is taken in the metric of $L^2(0, \infty)$, and

$$\|F\|^2 = 2\pi \|f\|_+^2 .$$

The subscript "+" indicates that the underlying Hilbert space is $L^2(0, \infty)$ rather than $L^2(-\infty, \infty)$. We also say that $f(x)$, $F(x)$ is a Mellin transform pair.

Theorem. *Let $p(x), q(x) \in L^2(0, \infty)$ and let $P(x), Q(x)$ be the corresponding Mellin transforms in $L^2(-\infty, \infty)$. Assume $P(x), Q(x) \in L^\infty(-\infty, \infty)$ and $P(x)$ is real valued (equivalently $\bar{p}(x) = x^{-1}p(x^{-1})$). For $x>0$, $t>0$ set*

$$L(x, t) = t^{-1}p(xt^{-1}) + 2\int_1^\infty q(t\lambda)\bar{q}(x\lambda)\,d\lambda.$$

(A) *If $f(x) \in L^2(0, \infty)$ then*

$$\int_0^\infty L(x, t)f(t)\,dt = \lim_{a\searrow 0} \int_a^{1/a} L(x, t)f(t)\,dt$$

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exists in the metric of $L^2(0, \infty)$, *and the linear operator defined in* $L^2(0, \infty)$ *by*

$$T: f(x) \to \int_0^\infty L(x, t) f(t)\, dt$$

is bounded and selfadjoint.

(B) *The operator* T *is unitarily equivalent to the singular integral operator* S *in* $L^2(-\infty, \infty)$ *defined by* (1) *with coefficients*

$$A(x) = P(x) + |Q(x)|^2, \quad K(x) = Q(x).$$

Proof. The Hilbert transformation in $L^2(-\infty, \infty)$ is defined by

$$H: F(x) \to (\pi i)^{-1}\, \mathrm{P.V.} \int_{-\infty}^{+\infty} (x-t)^{-1} F(t)\, dt.$$

Let J denote multiplication by the function

$$\sigma(x) = \begin{cases} 1, & 0 < x < 1 \\ -1, & 1 < x < \infty \end{cases}$$

in $L^2(0, \infty)$. For any Mellin transform pairs $f(x)$, $F(x)$ and $g(x)$, $G(x)$ we have then

$$\langle HF, G \rangle = 2\pi \langle Jf, g \rangle_+ .$$

See TITCHMARSH [8, Chapt. V].

If $f(x)$, $F(x)$ and $g(x)$, $G(x)$ are Mellin transform pairs and $F(x)$ is essentially bounded, then

$$h(x) = \int_0^\infty t^{-1} f(xt^{-1}) g(t)\, dt, \quad H(x) = F(x) G(x)$$

is a Mellin transform pair. See e.g. [8, p. 90].

Let $f(x)$, $F(x)$ and $g(x)$, $G(x)$ be Mellin transform pairs. It is easy to see that

$$B(f, g) = \int_0^\infty \left(\int_0^\infty t^{-1} p(xt^{-1}) f(t)\, dt \right) \bar{g}(x)\, dx + \\ + 2 \int_1^\infty \left(\int_0^\infty q(t\lambda) f(t)\, dt \right) \left(\int_0^\infty \bar{q}(x\lambda) \bar{g}(x)\, dx \right) d\lambda$$

defines a bounded symmetric bilinear form on $L^2(0, \infty)$. Therefore $B(f, g) = \langle Tf, g \rangle_+$ for some bounded selfadjoint operator T. If $f(x)$, $g(x)$ vanish outside a compact subinterval of $(0, \infty)$, then

$$\langle Tf, g \rangle_+ = \int_0^\infty \left(\int_0^\infty L(x, t) f(t)\, dt \right) \bar{g}(x)\, dx,$$

the rearrangement of integrals being justified by absolute convergence. From this we deduce (A).

In the same notation we have

$$\langle Tf, g\rangle_+ = \int_0^\infty\int_0^\infty t^{-1}p(xt^{-1})f(t)\bar{g}(x)\,dt\,dx + $$
$$+2\int_1^\infty\left(\int_0^\infty q(t\lambda)f(t)\,dt\right)\left(\int_0^\infty \bar{q}(x\lambda)\bar{g}(x)\,dx\right)d\lambda -$$
$$-\int_0^\infty\left(\int_0^\infty q(t\lambda)f(t)\,dt\right)\left(\int_0^\infty \bar{q}(x\lambda)\bar{g}(x)\,dx\right)d\lambda +$$
$$+\int_0^\infty\left(\int_0^\infty q(t\lambda)f(t)\,dt\right)\left(\int_0^\infty \bar{q}(x\lambda)\bar{g}(x)\,dx\right)d\lambda = \int_0^\infty\int_0^\infty t^{-1}p(xt^{-1})f(t)\bar{g}(x)\,dt\,dx +$$
$$+\int_0^\infty\left(\int_0^\infty q(t\lambda)f(t)\,dt\right)\left(\int_0^\infty \bar{q}(x\lambda)\bar{g}(x)\,dx\right)d\lambda -$$
$$-\int_0^\infty \sigma(\lambda)\left(\int_0^\infty q(t\lambda)f(t)\,dt\right)\left(\int_0^\infty \bar{q}(x\lambda)\bar{g}(x)\,dx\right)d\lambda =$$
$$= (2\pi)^{-1}\{\langle PF, G\rangle + \langle QF, QG\rangle + \langle HQF, QG\rangle\} = (2\pi)^{-1}\langle SF, G\rangle,$$

where S is the operator (1) with coefficients as in (B). The theorem follows.

The Hilbert matrix. If $p(x)=0$, $q(x)=\frac{1}{2}e^{-x/2}$, then

$$L(x, t) = (x+t)^{-1}e^{-(x+t)/2}.$$

In this case the matrix of T with respect to the orthonormal basis in $L^2(0, \infty)$ consisting of weighted Laguerre polynomials $\varphi_j(x)=L_j(x)e^{-x/2}$ $(j=0, 1, 2, \ldots)$ is the Hilbert matrix $\{1/(j+k+1)\}_{j,k=0}^{\infty}$. See ROSENBLUM [4, 7]. By our theorem T is unitarily equivalent to the operator S defined by (1) with coefficients

$$K(x) = 2^{-1/2+ix}\Gamma\left(\tfrac{1}{2}+ix\right), \quad A(x) = |K(x)|^2 = \tfrac{1}{2}\pi/\cosh \pi x.$$

From the theory of singular integral operators (PINCUS [2, 3], ROSENBLUM [6] we see at once that the spectrum of S is purely absolutely continuous, it consists of the interval $[0, \pi]$, and has uniform multiplicity 1 on the interval. In particular this gives a new proof of Hilbert's inequality

$$\left|\sum_{j,k=0}^{\infty} x_j\bar{x}_k/(j+k+1)\right| \leqq \pi\sum_{j=0}^{\infty}|x_j|^2$$

and the fact that π is the best constant (see HARDY, LITTLEWOOD, and PÓLYA [1, Chapt. IX]). ROSENBLUM [5] diagonalizes the Hilbert matrix by a different method. His method has the advantage that it can be adapted to the generalized Hilbert matrix $\{1/(j+k+1-\nu)\}_{j,k=0}^{\infty}$ where ν is a real parameter which is not a positive integer.

Miscellaneous examples. (1) If $p(x)=0$, $q(x)=2^{-\frac{1}{2}}(1+x)^{-1}$ then

$$L(x,t)=(x-t)^{-1}\log[(1+t^{-1})/(1+x^{-1})].$$

In the theorem

$$K(x)=2^{-1/2}\pi/\cosh\pi x,\quad A(x)=K(x)^2=\tfrac{1}{2}\pi^2/\cosh^2\pi x.$$

So T has absolutely continuous spectrum with uniform multiplicity 1 on $[0,\pi^2]$.

(2) If $p(x)=0$, $q(x)=2^{-1/2}e^{-x^2}$ then

$$L(x,t)=(x^2+t^2)^{-1/2}\operatorname{erfc}\left((x^2+t^2)^{1/2}\right)$$

where

$$\operatorname{erfc}x=\int_x^\infty e^{-t^2}\,dt,\quad x>0.$$

We have

$$K(x)=2^{-3/2}\Gamma\left(\tfrac{1}{4}+\tfrac{1}{2}ix\right),\quad A(x)=2^{-3}\left|\Gamma\left(\tfrac{1}{4}+\tfrac{1}{2}ix\right)\right|.^2$$

The operator T has absolutely continuous spectrum with multiplicity 1 on $[0,c]$ where

$$c=\Gamma(1/4)^2/4=3.28626\dots.$$

(3) If $p(x)=0$, $q(x)=2^{-\frac{1}{2}}(1+x^2)^{-1}$ then

$$L(x,t)=[x\operatorname{arc\,cot}x-t\operatorname{arc\,cot}t]/(x^2-t^2),$$

$$K(x)=\tfrac{1}{2}\pi/[\cosh(\tfrac{1}{2}\pi x)+i\sinh(\tfrac{1}{2}\pi x)],\quad A(x)=\tfrac{1}{4}\pi^2/\cosh(\pi x).$$

The operator T has absolutely continuous spectrum with multiplicity 1 on $[0,\frac{1}{2}\pi^2]$.

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