Truncated moment problems for operators

By T. ANDO in Sapporo (Japan)

Dedicated to Professor H. Nakano on his sixtieth birthday

1. Introduction

It has been well recognized that moment problems have close connection with spectral theory for selfadjoint or unitary operators. The trigonometric moment theory, for instance, is applied to establish the spectral theorem for unitary operators. while the spectral theory affords a unified treatment for various kinds of moment problems even for operators (see SZ.-NAGY [7], and KREĬN and KRASNOSELSKIĬ [5]), In this paper we shall further develop this approach to solve truncated moment problems for operators. Our basic problem is to find conditions for the existence of an increasing operator-valued function $E(\lambda)$ such that

$$\text{(power moment problem):} \quad A_k = \int_I \lambda^k \, dE(\lambda) \qquad (k=0,1,\ldots,m)$$

or

$$\text{(trigonometric moment problem):} \quad A_k = \int_I e^{ik\lambda} dE(\lambda) \qquad (k=0,1,\ldots,m),$$

where $A_0, \ldots, A_m$ are given bounded linear operators on a Hilbert space and I is a finite or infinite interval. While the classical moment problems, in which each A_k is an operator of rank one, have been settled mostly with help of function theory and an extension theorem for positive linear functionals (see AHIEZER [1], and AHIEZER and KREĬN [2]), the key to our development is provided with extension theorems for symmetric operators with non-dense domain, as shown by KREĬN and KRASNOSELSKIĬ [5] on proving the power moment problem with $m=2n$ and $I=[-1,1]$.

2. Preliminaries

All operators are assumed to be linear. To a bounded operator T from a Hilbert space $\mathfrak{K}_1$ to a Hilbert space $\mathfrak{K}_2$, there corresponds its adjoint T^* as a bounded operator from K_2 to K_1. The square root of T^*T will be denoted by $|T|$. A *positive* operator means a non-negative selfadjoint operator. The *inverse* A^{-1} of a bounded positive operator A is, by definition, the uniquely determined operator with $\mathfrak{D}(A^{-1})=\mathfrak{R}(A)$ and $A^{-1}A=1-P$, where P is the orthogonal projection onto the kernel of A. $A^{-1/2}$ stands for the inverse of $A^{1/2}$. An operator S on a Hilbert space is *symmetric* if $(Sf, g)=(f, Sg)$ for $f, g\in\mathfrak{D}(S)$.

A bounded positive operator A on a Hilbert space $\mathfrak{K}$ gives rise to a new semi-inner product: $(f, g)_A=(Af, g)$. The completion of $\mathfrak{K}$ with respect to this semi-inner product will be denoted by $\mathfrak{K}_A$. A bounded positive operator C is called *A-bounded* (resp. *A-closable*), if the canonical identification of $\mathfrak{K}$ is a bounded (resp. closable) operator from $\mathfrak{K}_A$ to $\mathfrak{K}_C$. Thus C is A-bounded (resp. A-closable) if and only if there exists a bounded (resp. closed) operator X with $C=|XA^{1/2}|^2$.

Lemma. *Let A be a bounded positive operator. Then a vector h belongs to $\mathfrak{D}(A^{-\frac{1}{2}})$ if and only if, with the convention* $0/0=0$,

$$\sup_{f\in\mathfrak{D}(A)}\frac{|(f, h)|^2}{(Af, f)}<\infty.$$

In this case the left hand side coincides with $\|A^{-1/2}h\|^2$.

Proof. Suppose that h belongs to $\mathfrak{D}(A^{-1/2})$. Then

$$\frac{|(f, h)|^2}{(Af, f)}=\frac{|(A^{1/2}f, A^{-1/2}h)|^2}{\|A^{1/2}f\|^2}.$$

Since $\mathfrak{R}(A^{1/2})$ is dense in the orthogonal complement of the kernel of A,

$$\sup_{f\in\mathfrak{D}(A)}\frac{|(f, h)|^2}{(Af, f)}=\|A^{-1/2}h\|^2.$$

Suppose conversely that the left hand side is finite. Then h is orthogonal to the kernel of A. Let $E(\lambda)$ $(0\leqq\lambda<\infty)$ be the resolution of identity for A. Then it follows that

$$h=\lim_{\varepsilon\to 0}\big(1-E(\varepsilon)\big)h.$$

Since each $\big(1-E(\varepsilon)\big)h$ belongs to $\mathfrak{D}(A^{-1/2})$, the closedness of $A^{-1/2}$ implies that h belongs to $\mathfrak{D}(A^{-1/2})$ if and only if

$$\sup_{\varepsilon>0}\|A^{-1/2}\big(1-E(\varepsilon)\big)h\|<\infty.$$

Now since

$$\frac{|(g,(1-E(\varepsilon))h)|^2}{(Ag,g)} \leqq \frac{|((1-E(\varepsilon))g,h)|^2}{(A(1-E(\varepsilon))g,(1-E(\varepsilon))g)} \leqq \sup_{f\in\mathfrak{D}(A)} \frac{|(f,h)|^2}{(Af,f)},$$

it follows from the preceding part of the proof that

$$\sup_{\varepsilon>0} \|A^{-1/2}(1-E(\varepsilon))h\|^2 = \sup_{\varepsilon>0}\sup_{g\in\mathfrak{D}(A)} \frac{|(g,(1-E(\varepsilon))h)|^2}{(Ag,g)} \leqq \sup_{f\in\mathfrak{D}(A)} \frac{|(f,h)|^2}{(Af,f)} < \infty.$$

This completes the proof.

Each bounded operator T from a direct sum $\bigoplus_{j=1}^{m} \mathfrak{K}_j$ to another $\bigoplus_{i=1}^{n} \mathfrak{K}_i$ is represented in the natural way by the corresponding $n\times m$ matrix (T_{ij}), where T_{ij} is a bounded operator from $\mathfrak{K}_j$ to $\mathfrak{K}_i$.

Proposition 1. *Let A be a bounded positive operator on $\mathfrak{K}_1$, and B a bounded operator from $\mathfrak{K}_2$ to $\mathfrak{K}_1$. There exists a bounded operator C on $\mathfrak{K}_2$, for which the matrix $\begin{bmatrix} A & B \\ B^* & C\end{bmatrix}$ is positive, if and only if $|B^*|^2$ is A-bounded. When this condition is fulfilled, $|A^{-1/2}B|^2$ is the minimum of all such C.*

Proof (cf. [3]). A matrix $\begin{bmatrix} A & B \\ B^* & C\end{bmatrix}$ is positive, if and only if

$$(Af,f)+(Bg,f)+(B^*f,g,)+(Cg,g)\geqq 0 \qquad (f\in\mathfrak{K}_1,\ g\in\mathfrak{K}_2),$$

that is,

$$|(Bg,f)|^2\leqq (Af,f)(Cg,g) \qquad (f\in\mathfrak{K}_1,\ g\in\mathfrak{K}_2).$$

By Lemma this last condition can be converted to the inequality

$$|A^{-1/2}B|^2 \leqq C$$

and the minimum property of $|A^{-1/2}B|^2$ is clear. On the other hand, $|B^*|^2$ is A-bounded if and only if for some $\gamma>0$

$$\|B^*f\|^2\leqq \gamma(Af,f) \qquad (f\in\mathfrak{K}_1),$$

or, with $C=\gamma$,

$$|(Bg,f)|^2\leqq (Af,f)(Cg,g) \qquad (f\in\mathfrak{K}_1,\ g\in\mathfrak{K}_2).$$

This completes the proof.

Corollary 1. *If a matrix $T=\begin{bmatrix} A & B \\ B^* & C\end{bmatrix}$ is positive, then*

$$J^*PJ = \begin{bmatrix} A & B \\ B^* & |A^{-1/2}B|^2\end{bmatrix},$$

where J is the canonical map of $\mathfrak{K}=\mathfrak{K}_1\oplus\mathfrak{K}_2$ to $\mathfrak{K}_T$ and P is the orthogonal projection of $\mathfrak{K}_T$ onto the closure of $J(\mathfrak{K}_1)$. Thus $J(\mathfrak{K}_1)$ is dense in $\mathfrak{K}_T$ if and only if $C=|A^{-1/2}B|^2$.

Proof. Since $PJf=Jf$ for $f\in\mathfrak{K}_1$, it follows

$$(J^*PJf,f)=(Af,f) \quad \text{and} \quad (J^*PJf,g)=(B^*f,g) \qquad (f\in\mathfrak{K}_1,\, g\in\mathfrak{K}_2).$$

Finally it results

$$(J^*PJg,g)=(PJg,PJg)_T=\sup_{f\in\mathfrak{K}_1}\frac{|(Jg,Jf)_T|^2}{(Jf,Jf)_T}=\sup_{f\in\mathfrak{K}_1}\frac{|(Bg,f)|^2}{(Af,f)}=\|A^{-1/2}Bg\|^2.$$

Corollary 2. *A symmetric operator T on a Hilbert space $\mathfrak{K}$ admits a positive extension of norm $\leqq 1$ if and only if*

$$\|Tf\|^2\leqq(Tf,f) \qquad (f\in\mathfrak{D}(A)).$$

Proof. The inequality is obviously satisfied if T admits the required extension. If, conversely, the inequality is valid, it follows $\|Tf\|\leqq\|f\|$, so that the domain $\mathfrak{D}(T)$ may be assumed closed. Let $\mathfrak{K}_1=\mathfrak{D}(T)$, $\mathfrak{K}_2=\mathfrak{K}_1^{\perp}$, $A=QTQ$ and $B^*=(1-Q)TQ$, where Q is the orthogonal projection onto $\mathfrak{K}_1$. Since the inequality in question is equivalent to

$$|(Tf,h)|^2\leqq(Tf,f)(h,h) \qquad (f\in\mathfrak{K}_1,\, h\in\mathfrak{K}),$$

$|B^*|^2$ is A-bounded, consequently by Corollary 1 the operator $\hat{T}=\begin{bmatrix} A & B \\ B^* & |A^{-1/2}B|^2\end{bmatrix}$ gives a positive extension of T. The assertion on norm follows from the relation

$$(\hat{T}h,h)=\sup_{f\in\mathfrak{D}(T)}\frac{|(Tf,h)|^2}{(Tf,f)}\leqq(h,h).$$

Remark 1. Corollary 2 is just a variant of the fundamental theorem of Kreĭn (see [5; § 6] or [6; n° 125]) that a symmetric operator of norm $\leqq 1$ admits a selfadjoint extension of norm $\leqq 1$. Corollary 1 corresponds to Lemma 7. 1 in [5].

$E(\lambda)$ $(a\leqq\lambda\leqq b)$ is called a *spectral function*, if (1) each $E(\lambda)$ is a bounded positive operator, (2) $E(\lambda)\leqq E(\mu)$ for $\lambda\leqq\mu$, (3) $E(\lambda+0)=E(\lambda)$ and (3′) $E(a+0)=0$ and/or $E(b-0)=E(b)$ in case $a=-\infty$ and/or $b=\infty$. It is *orthogonal*, if each $E(\lambda)$ is an orthogonal projection. The basic connection between spectral functions and orthogonal spectral functions was established by Naĭmark (see [7]). We shall use his result in the following modified form: if $E(\lambda)$ is a spectral function in a Hilbert space $\mathfrak{K}$, there exist a Hilbert space $\hat{\mathfrak{K}}$, an orthogonal spectral function $P(\lambda)$ in $\hat{\mathfrak{K}}$ and a bounded operator J from $\mathfrak{K}$ to $\hat{\mathfrak{K}}$ such that $E(\lambda)=J^*P(\lambda)J$.

Given a spectral function $E(\lambda)$ $(a\leqq\lambda\leqq b)$, the *weak integral* $\int_a^b \lambda^k dE(\lambda)$ determines a closed symmetric operator with dense domain $(k=0,1,\ldots)$. When $E(\lambda)$ is an orthogonal spectral function, the domain of the operator $\int_a^b \lambda^k dE(\lambda)$ consists of vectors f, for which the strong integral $\int_a^b \lambda^k dE(\lambda) f$ converges.

Proposition 2. *If S is a closable symmetric operator on a Hilbert space $\mathfrak{R}$, there exists a spectral function $E(\lambda)$ $(-\infty \leqq \lambda \leqq \infty)$ such that, for all k, j,*

$$(S^k f, S^j g) = \int_{-\infty}^{\infty} \lambda^{j+k} \, d(E(\lambda) f, g) \quad (f \in \mathfrak{D}(S^k), g \in \mathfrak{D}(S^j)).$$

Proof. The Krasnoselskiĭ Theorem [4; §3] on closed symmetric operators guarantees that there exists a selfadjoint operator $\hat{S}$ on the direct sum of $\mathfrak{R}$ and its copy, which is an extension of S. Then the spectral function $E(\lambda)$, defined by

$$E(\lambda) f = QP(\lambda) f \qquad (f \in \mathfrak{R}),$$

meets the requirement, where $P(\lambda)$ is the resolution of identity for $\hat{S}$ and Q is the orthogonal projection onto $\mathfrak{R}$. In fact, for $f \in \mathfrak{D}(S^k)$ and $g \in \mathfrak{D}(S^j)$,

$$(S^k f, S^j g) = (\hat{S}^k f, \hat{S}^j g) = \int_{-\infty}^{\infty} \lambda^{j+k} \, d(P(\lambda) f, g) = \int_{-\infty}^{\infty} \lambda^{j+k} \, d(E(\lambda) f, g).$$

If V is an isometric operator with non-dense domain in a Hilbert space $\mathfrak{R}$, it admits obviously a unitary extension $\hat{V}$ on the direct sum of $\mathfrak{R}$ and its copy. Then, as in the proof of Proposition 2, there exists a spectral function $E(\lambda)$ $(-\pi \leqq \lambda \leqq \pi)$ such that $\int_{-\pi}^{\pi} e^{ik\lambda} dE(\lambda)$ is an extension of V^k for all k.

Proposition 3. *An isometric operator V with non-dense domain admits the representation for some spectral function $E(\lambda)$ $(-\pi/2 \leqq \lambda \leqq \pi/2)$:*

$$V^k f = \int_{-\pi/2}^{\pi/2} e^{ik\lambda} \, dE(\lambda) f \qquad (f \in \mathfrak{D}(V^k),\ k = 0, 1, \ldots),$$

if and only if

$$\operatorname{Re}(Vf, f) \geqq 0 \qquad (f \in \mathfrak{D}(V)).$$

Proof. If V admits such representation, it follows

$$\operatorname{Re}(Vf, f) = \int_{-\pi/2}^{\pi/2} \cos\lambda \, d(E(\lambda) f, f) \geqq 0.$$

Suppose, conversely, that the inequality is satisfied. Since $(V+1)h = 0$ implies $h=0$ by assumption, the operator

$$T = \frac{i+1}{2}(1-iV)(V+1)^{-1}$$

is symmetric. Calculation shows that, with $(V+1)f = g$,

$$(Tg, g) = (f,f) + \mathrm{Re}\,(Vf,f) + \mathrm{Im}\,(Vf,f)$$

and

$$(Tg, Tg) = \mathrm{Im}\,(Vf,f) + (f,f),$$

hence

$$\|Tg\|^2 \leqq (Tg, g).$$

Then by Corollary 2 T admits a selfadjoint extension $0 \leqq \hat{T} \leqq 1$. The operator

$$\hat{V} = \left(\frac{1+i}{2} - \hat{T}\right)\left(\hat{T} - \frac{1-i}{2}\right)^{-1}$$

is a unitary extension of V with $\mathrm{Re}\,(\hat{V}) \geqq 0$. Since the spectrum of V is concentrated on the arc $\{e^{i\lambda}: -\pi/2 \leqq \lambda \leqq \pi/2\}$, the resolution of identity for $\hat{V}$ produces a spectral function of the kind we expected.

Remark 2. Proposition 3 can be modified to get a condition for a representation of the form $\int_{-\tau}^{\tau} e^{ik\lambda} dE(\lambda)$ for some $0 < \tau < \pi$.

3. Hamburger truncated moment problem

A finite sequence $\{A_0, \ldots, A_m\}$ of bounded selfadjoint operators on a Hilbert space $\mathfrak{H}$ is called a *Hamburger moment sequence*, if there exists a spectral function $E(\lambda)$ $(-\infty \leqq \lambda \leqq \infty)$ such that

$$A_k = \int_{-\infty}^{\infty} \lambda^k \, dE(\lambda) \qquad (k = 0, 1, \ldots, m).$$

The direct sum of $n+1$ copies of $\mathfrak{H}$ will be denoted simply by $\bigoplus_0^n \mathfrak{H}$ and a general element in it by a row vector $\langle f_0, \ldots, f_n \rangle$. $\bigoplus_0^n \mathfrak{H}$ $(n \leqq m)$ is identified with the subspace of $\bigoplus_0^m \mathfrak{H}$, consisting of vectors of the form $\langle f_0, \ldots, f_n, 0, \ldots, 0 \rangle$.

For a sequence $\{A_0, A_1, \ldots\}$ of bounded selfadjoint operators, the n-th *Hankel matrix* $\mathbf{H}_n$ is, by definition, the operator on $\bigoplus_0^n \mathfrak{K}$, given by

$$\mathbf{H}_n = \begin{bmatrix} A_0 & A_1 & A_2 \ldots A_n \\ A_1 & A_2 & A_3 \ldots A_{n+1} \\ \ldots & \ldots & \ldots \\ A_n & A_{n+1} & \ldots A_{2n} \end{bmatrix}.$$

The n-th Hankel matrices for $\{A_1, A_2, \ldots\}$ and $\{A_2, A_3, \ldots\}$ will be denoted by $\mathbf{K}_n$ and $\mathbf{L}_n$ respectively, i.e.

$$\mathbf{K}_n = \begin{bmatrix} A_1 & A_2 & \ldots A_{n+1} \\ A_2 & A_3 & \ldots A_{n+2} \\ \ldots & \ldots & \ldots \\ A_{n+1} & A_{n+2} & \ldots A_{2n+1} \end{bmatrix} \quad \text{and} \quad \mathbf{L}_n = \begin{bmatrix} A_2 & A_3 & \ldots A_{n+2} \\ A_3 & A_4 & \ldots A_{n+3} \\ \ldots & \ldots & \ldots \\ A_{n+2} & A_{n+3} & \ldots A_{2n+2} \end{bmatrix}.$$

The n-th *marginal matrix* $\mathbf{B}_n$ for $\{A_0, A_1, \ldots\}$ is, by definition, the operator from $\mathfrak{K}$ to $\bigoplus_0^n \mathfrak{K}$ given by

$$\mathbf{B}_n = \begin{bmatrix} A_{n+1} \\ A_{n+2} \\ \vdots \\ A_{2n+1} \end{bmatrix}.$$

The n-th marginal matrices for $\{A_1, A_2, \ldots\}$ and $\{A_2, A_4, \ldots\}$ will be denoted by $\mathbf{C}_n$ and $\mathbf{D}_n$, respectively, i.e.

$$\mathbf{C}_n = \begin{bmatrix} A_{n+2} \\ A_{n+3} \\ \vdots \\ A_{2n+2} \end{bmatrix} \quad \text{and} \quad \mathbf{D}_n = \begin{bmatrix} A_{n+3} \\ A_{n+4} \\ \vdots \\ A_{2n+3} \end{bmatrix}.$$

Theorem 1. *$\{A_0, \ldots, A_{2n+2}\}$ is a Hamburger moment sequence if and only if the Hankel matrix $\mathbf{H}_{n+1}$ is positive and there exists a closed operator X such that*

$$A_{2n+2} = A + |XC^{1/2}|^2,$$

where

$$A = |\mathbf{H}_n^{-1/2}\mathbf{B}_n|^2 \quad \textit{and} \quad C = A_{2n} + A - |(\mathbf{H}_{n-1} + \mathbf{L}_{n-1})^{-1/2}(\mathbf{B}_{n-1} + \mathbf{D}_{n-1})|^2.$$

Proof. Suppose that $\{A_0, \ldots, A_{2n+2}\}$ is a Hamburger moment sequence with respect to a spectral function $E(\lambda)$. In view of the Naĭmark theorem there exist an orthogonal spectral function $P(\lambda)$ in a Hilbert space $\hat{\mathfrak{K}}$ and a bounded

J from $\mathfrak{K}$ to $\hat{\mathfrak{K}}$ such that $E(\lambda)=J^*P(\lambda)J$. Consider the selfadjoint operator

$$\hat{A} = \int_{-\infty}^{\infty} \lambda\, dP(\lambda).$$

Since Jf belongs to the domain of $\hat{A}^{n+1}$ for all $f\in\mathfrak{K}$, it follows

$$\sum_{j,k=0}^{n+1} (A_{j+k}f_k, f_j) = \left\|\sum_{k=0}^{n+1} \hat{A}^k\cdot Jf_k\right\|^2 \geqq 0,$$

hence the Hankel matrix $\mathbf{H}_{n+1}$ is positive. Since

$$\|\langle f_0, \cdots, f_{n+1}\rangle\|_{\mathbf{H}_{n+1}} = \left\|\sum_{k=0}^{n+1} \hat{A}^k\cdot Jf_k\right\|,$$

the shift operator $\mathbf{S}$:

$$\mathbf{S}\langle f_0, \ldots, f_n, 0\rangle = \langle 0, f_0, \ldots, f_n\rangle$$

is well-defined and closable in $(\bigoplus_0^{n+1}\mathfrak{K})_{\mathbf{H}_{n+1}}$ because of the closedness of $\hat{A}$. $\mathbf{S}$ is obviously symmetric. Consider further the truncated shift operator: $\mathbf{T}=\mathbf{PSP}$, where $\mathbf{P}$ is the orthogonal projection onto the closure of the canonical image of $\bigoplus_0^n\mathfrak{K}$ in $(\bigoplus_0^{n+1}\mathfrak{K})_{\mathbf{H}_{n+1}}$. $\mathbf{T}$ is also symmetric and closable, because it is densely defined. It follows from Corollary 1 that

$$\|\mathbf{T}\langle f_0, \cdots, f_n, 0\rangle\|^2_{\mathbf{H}_{n+1}} = \|\mathbf{P}\langle 0, f_0, \cdots, f_n\rangle\|^2_{\mathbf{H}_{n+1}} =$$
$$= \|\langle 0, f_0, \cdots, f_n\rangle\|^2_{\mathbf{H}_{n+1}} - ((A_{2n+2}-A)f_n, f_n) =$$
$$= \|\mathbf{S}\langle f_0, \cdots, f_n, 0\rangle\|^2_{\mathbf{H}_{n+1}} - ((A_{2n+2}-A)f_n, f_n).$$

Suppose that

$$\|\langle f_0^{(N)}, \cdots, f_n^{(N)}\rangle\|^2_{\mathbf{H}_n} + \|\mathbf{T}\langle f_0^{(N)}, \cdots, f_n^{(N)}, 0\rangle\|^2_{\mathbf{H}_{n+1}} \to 0$$

and

$$\|f_n^{(N)} - f_n^{(M)}\|^2_{A_{n+2}-A} \to 0 \quad (N, M\to\infty).$$

Then

$$\|\mathbf{S}\langle f_0^{(N)}, \cdots, f_n^{(N)}, 0\rangle - \mathbf{S}\langle f_0^{(M)}, \cdots, f_n^{(M)}, 0\rangle\|^2_{\mathbf{H}_{n+1}} =$$
$$= \|\mathbf{T}\langle f_0^{(N)}, \cdots, f_n^{(N)}, 0\rangle - \mathbf{T}\langle f_0^{(M)}, \cdots, f_n^{(M)}, 0\rangle\|^2_{\mathbf{H}_{n+1}} +$$
$$+ \|f_n^{(N)} - f_n^{(M)}\|^2_{A_{2n+2}-A} \to 0 \quad (N, M\to\infty).$$

Since both $\mathbf{S}$ and $\mathbf{T}$ are closable, it follows that

$$\|f_n^{(N)}\|^2_{A_{2n+2}-A} = \|\mathbf{S}\langle f_0^{(N)}, \ldots, f_n^{(N)}, 0\rangle\|^2_{\mathbf{H}_{2n+1}} - \|\mathbf{T}\langle f_0^{(N)}, \cdots, f_n^{(N)}, 0\rangle\|^2_{\mathbf{H}_{2n+1}} \to 0 \quad (N\to\infty).$$

Let $\mathbf{L}_n'$ be the n-th Hankel matrix for $\{A_2, \ldots, A_{2n+1}, A\}$ and let $\mathbf{Q}$ be the orthogonal projection of $(\bigoplus_0^n\mathfrak{K})_{\mathbf{H}_n+\mathbf{L}_n'}$ onto the closure of the canonical image of $\bigoplus_0^{n-1}\mathfrak{K}$. Then

$$\|\langle f_0, \cdots, f_n\rangle\|^2_{\mathbf{H}_n+\mathbf{L}_n'} = \|\langle f_0, \cdots, f_n\rangle\|^2_{\mathbf{H}_n} + \|\mathbf{T}\langle f_0, \ldots, f_n, 0\rangle\|^2_{\mathbf{H}_{n+1}}.$$

Now the above convergence implication can be written in the form

$$\|(1-\mathbf{Q})\langle 0, \dots, 0, f^{(N)}\rangle\|^2_{\mathbf{H}_n+\mathbf{L}_n'} \to 0$$

and

$$\|f^{(N)}-f^{(M)}\|^2_{A_{2n+2}-A} \to 0 \quad (N, M \to \infty)$$

implies

$$\|f^{(N)}\|^2_{A_{2n+2}-A} \to 0 \quad (N \to \infty),$$

for, by the definition of the projection $\mathbf{Q}$, there are $f_0^{(N)}, \dots, f_{n-1}^{(N)}$ such that

$$\|\langle f_0^{(N)}, \dots, f_{n-1}^{(N)}, f^{(N)}\rangle\|_{\mathbf{H}_n+\mathbf{L}_n'} \leqq \|(1-\mathbf{Q})\langle 0, \dots, 0, f^{(N)}\rangle\|_{\mathbf{H}_n+\mathbf{L}_n'}+1/N.$$

This means, however, that $A_{2n+2}-A$ is C-closable, for it follows from Corollary 1 that

$$\|(1-\mathbf{Q})\langle 0, \dots, 0, f\rangle\|^2_{\mathbf{H}_n+\mathbf{L}_n'} =$$
$$= ((A_{2n}+A)f, f)-(|(\mathbf{H}_{n-1}+\mathbf{L}_{n-1})^{-1/2}(\mathbf{B}_{n-1}+\mathbf{D}_{n-1})|^2 f, f) = (Cf, f).$$

Suppose, conversely, that $\mathbf{H}_{n+1}$ is positive and $A_{2n+2}-A$ is C-closable. The above argument can be traced backwards to conclude that the shift operator $\mathbf{S}$ is symmetric and closable in $(\bigoplus_0^{n+1} \mathfrak{K})_{\mathbf{H}_{n+1}}$. Let $\mathbf{E}(\lambda)$ be the spectral function in $(\bigoplus_0^{n+1} \mathfrak{K})_{\mathbf{H}_{n+1}}$ for $\mathbf{S}$, guaranteed by Proposition 2. Then it follows, for $j, k \leqq n+1$,

$$(A_{j+k}f, g) = (\mathbf{S}^k\langle f, 0, \dots, 0\rangle, \mathbf{S}^j\langle g, 0, \dots, 0\rangle)_{\mathbf{H}_{n+1}} =$$
$$= \int_{-\infty}^{\infty} \lambda^{j+k}\, d(\mathbf{E}(\lambda)\langle f, 0, \dots, 0\rangle, \langle g, 0, \dots, 0\rangle)_{\mathbf{H}_{n+1}},$$

hence the expected spectral function $E(\lambda)$ can be given by

$$(E(\lambda)f, g) = (\mathbf{E}(\lambda)\langle f, 0, \dots, 0\rangle, \langle g, 0, \dots, 0\rangle)_{\mathbf{H}_{n+1}}.$$

This completes the proof.

Corollary 3. *If the Hankel matrix $\mathbf{H}_{n+1}$ is positive, there exists a selfadjoint operator A such that $0 \leqq A \leqq A_{2n+2}$ and the sequence $\{A_0, \dots, A_{2n+1}, A\}$ is a Hamburger moment sequence.*

Proof. Since $\mathbf{H}_{n+1}$ is positive, by Proposition 1

$$A = |\mathbf{H}_n^{-1/2}\mathbf{B}_n|^2$$

is bounded. Then, with $X=0$, the conditions of Theorem 1 are fulfilled.

Corollary 4. *If the Hankel matrix $\mathbf{H}_{n+1}$ is positive and has a bounded inverse, then the sequence $\{A_0, \dots, A_{2n+2}, A_{2n+3}\}$ is a Hamburger moment sequence.*

Proof. Since $|\mathbf{B}_{n+1}^*|^2$ is obviously $\mathbf{H}_{n+1}$-bounded, the Hankel matrix $\mathbf{H}_{n+2}$, with A_{2n+4} replaced by $|\mathbf{H}_{n+1}^{-1/2}\mathbf{B}_{n+1}|^2$, is positive by Proposition 1. Now the assertion follows from Corollary 3.

Corollary 5. *$\{A_0, \ldots, A_{2n+1}\}$ is a Hamburger moment sequence if the Hankel matrix $\mathbf{H}_n$ is positive, $|\mathbf{C}_{n-1}^*|^2$ is $\mathbf{H}_{n-1}$-bounded, and there exists a bounded operator X such that*

$$A_{2n+1} = B + (A_{2n} - A)^{1/2} X,$$

where

$$A = |\mathbf{H}_{n-1}^{-1/2}\mathbf{B}_{n-1}|^2 \quad \text{and} \quad B = (\mathbf{H}_{n-1}^{-1/2}\mathbf{B}_{n-1})^*(\mathbf{H}_{n-1}^{-1/2}\mathbf{C}_{n-1}).$$

Proof. In view of Corollary 3, it suffices to show that the Hankel matrix $\mathbf{H}_{n+1}$ is positive for suitably chosen A_{2n+2}. However, by Proposition 1 the positivity of $\mathbf{H}_{n+1}$ is equivalent to the inequality:

$$|\mathbf{H}_{n-1}^{-1/2}[\mathbf{B}_{n-1}, \mathbf{C}_{n-1}]|^2 \leqq \begin{bmatrix} A_{2n}, & A_{2n+1} \\ A_{2n+1}, & A_{2n+2} \end{bmatrix},$$

or, with $C = |\mathbf{H}_{n-1}^{-1/2}\mathbf{C}_{n-1}|^2$,

$$\begin{bmatrix} A_{2n} - A, & A_{2n+1} - B \\ A_{2n+1} - B^*, & A_{2n+2} - C \end{bmatrix} \geqq 0.$$

By Proposition 1 the last inequality means that $|A_{2n+1} - B^*|^2$ is $(A_{2n} - A)$-bounded, what is guaranteed by the existence of X in the assertion of the corollary.

4. Stieltjes truncated moment problem

A finite sequence $\{A_0, \ldots, A_m\}$ of bounded positive operators is called a *Stieltjes moment sequence*, if there exists a spectral function $E(\lambda)$ $(0 \leqq \lambda \leqq \infty)$ such that

$$A_k = \int_0^\infty \lambda^k \, dE(\lambda) \qquad (k = 0, 1, \ldots, m).$$

To a spectral function $E(\lambda)$ $(0 \leqq \lambda \leqq \infty)$, there corresponds a spectral function $F(\lambda)$ $(-\infty \leqq \lambda \leqq \infty)$ such that

$$\int_{-\infty}^\infty \lambda^{2k} \, dF(\lambda) = \int_0^\infty \lambda^k \, dE(\lambda)$$

and

$$\int_{-\infty}^\infty \lambda^{2k+1} \, dF(\lambda) = 0 \qquad (k = 0, 1, \ldots).$$

In fact, $F(\lambda)$ can be given, for instance, by

$$F(\lambda) = \begin{cases} \frac{1}{2}E(\infty) - \frac{1}{2}E(\lambda^2 - 0) & \text{if } \lambda < 0, \\ \frac{1}{2}E(\lambda^2) + \frac{1}{2}E(\infty) & \text{if } \lambda \geqq 0. \end{cases}$$

Conversely, to a spectral function $F(\lambda)$ $(-\infty \leqq \lambda \leqq \infty)$ there corresponds a spectral function $E(\lambda)$ $(0 \leqq \lambda \leqq \infty)$ such that

$$\int_0^\infty \lambda^k \, dE(\lambda) = \int_{-\infty}^\infty \lambda^{2k} \, dF(\lambda) \qquad (k = 0, 1, \ldots).$$

Theorem 2. *$\{A_0, \ldots, A_{2n}\}$ is a Stieltjes moment sequence if and only if the Hankel matrices $\mathbf{H}_n$ and $\mathbf{K}_{n-1}$ are positive, and there exists a closed operator X such that*

$$A_{2n} = A + |XC^{1/2}|^2,$$

where

$$A = |\mathbf{H}_{n-1}^{-1/2} \mathbf{B}_{n-1}|^2 \quad \textit{and} \quad C = A_{2n-1} + A - |(\mathbf{K}_{n-2} + \mathbf{L}_{n-2})^{-1/2} (\mathbf{C}_{n-2} + \mathbf{D}_{n-2})|^2.$$

Proof. As remarked above, $\{A_0, \ldots, A_{2n}\}$ is a Stieltjes moment sequence if and only if $\{A_0, 0, A_1, 0, A_2, 0, \ldots, A_{2n-1}, 0, A_{2n}\}$ is a Hamburger moment sequence. The simultaneous row-column permutation:

$$2k \to k \; (k = 0, 1, \ldots, n-1) \quad \text{and} \quad 2k-1 \to n+k-1 \; (k = 1, \ldots, n)$$

brings the $2n$-th Hankel matrix for $\{A_0, 0, A_1, \ldots, 0, A_{2n}\}$ to the form

$$\begin{bmatrix} \mathbf{H}_{n-1} & 0 & \mathbf{B}_{n-1} \\ 0 & \mathbf{K}_{n-1} & 0 \\ \mathbf{B}_{n-1}^* & 0 & A_{2n} \end{bmatrix}.$$

Under the same permutation the $(2n-2)$-th Hankel matrix for $\{A_1, 0, \ldots, 0, A_{2n}\}$ is transformed to

$$\begin{bmatrix} \mathbf{K}_{n-2} & 0 \\ 0 & \mathbf{L}_{n-2} \end{bmatrix}.$$

Similarly the $(2n-2)$-th marginal matrices for $\{A_0, 0, A_1, \ldots, 0, A_{2n}\}$ and $\{A_1, 0, A_2, \ldots, 0, A_{2n}\}$ are transformed respectively to

$$\begin{bmatrix} 0 \\ \mathbf{C}_{n-2} \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ \mathbf{D}_{n-2} \end{bmatrix}.$$

Now the conditions of Theorem 1 for $\{A_0, 0, A_1, \ldots, 0, A_{2n}\}$ to be a Hamburger moment sequence are converted to the ones in the assertion of the present theorem, for the obvious permutation brings

$$\begin{bmatrix} \mathbf{H}_{n-1} & 0 & \mathbf{B}_{n-1} \\ 0 & \mathbf{K}_{n-1} & 0 \\ \mathbf{B}_{n-1}^* & 0 & A_{2n} \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} \mathbf{H}_n & 0 \\ 0 & \mathbf{K}_{n-1} \end{bmatrix},$$

and the following relations hold:

$$\left| \begin{bmatrix} \mathbf{H}_{n-1} & 0 \\ 0 & \mathbf{K}_{n-1} \end{bmatrix}^{-1/2} \begin{bmatrix} \mathbf{B}_{n-1} \\ 0 \end{bmatrix} \right|^2 = |\mathbf{H}_{n-1}^{-1/2} \mathbf{B}_{n-1}|^2$$

and

$$\left| \left(\begin{bmatrix} \mathbf{H}_{n-1} & 0 \\ 0 & \mathbf{K}_{n-2} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{n-2} & 0 \\ 0 & \mathbf{L}_{n-2} \end{bmatrix} \right)^{-1/2} \left(\begin{bmatrix} 0 \\ \mathbf{C}_{n-2} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{D}_{n-2} \end{bmatrix} \right) \right|^2 =$$

$$= |(\mathbf{K}_{n-2} + \mathbf{L}_{n-2})^{-1/2} (\mathbf{C}_{n-2} + \mathbf{D}_{n-2})|^2.$$

This completes the proof.

In the similar way the following theorem and the corollary can be derived from the results of the preceding section.

Theorem 2′. *$\{A_0, \ldots, A_{2n+1}\}$ is a Stieltjes moment sequence, if and only if the Hankel matrices $\mathbf{H}_n$ and $\mathbf{L}_n$ are positive and there exists a closed operator X such that*

$$A_{2n+1} = A + |XC^{1/2}|^2,$$

where

$$A = |\mathbf{K}_{n-1}^{-1/2} \mathbf{C}_{n-1}|^2 \quad \text{and} \quad C = A_{2n} + A - |(\mathbf{H}_{n-1} + \mathbf{K}_{n-1})^{-1/2} (\mathbf{B}_{n-1} + \mathbf{C}_{n-1})|^2.$$

Corollary 6. *If both the Hankel matrices $\mathbf{H}_n$ and $\mathbf{K}_{n-1}$ (resp. $\mathbf{K}_n$) are positive, there exists a bounded selfadjoint operator A such that $0 \leqq A \leqq A_{2n}$ (resp. $\leqq A_{2n+1}$) and the sequence $\{A_0, \ldots, A_{2n-1}, A\}$ (resp. $\{A_0, \ldots, A_{2n}, A\}$) is a Stieltjes moment sequence.*

5. Hausdorff truncated moment problem

A finite sequence $\{A_0, \ldots, A_m\}$ of bounded selfadjoint operators in a Hilbert space $\mathfrak{K}$ is called a *Hausdorff moment sequence*, if there exists a spectral function $E(\lambda)$ $(0 \leqq \lambda \leqq 1)$ such that

$$A_k = \int_0^1 \lambda^k \, dE(\lambda) \qquad (k = 0, 1, \ldots, m).$$

Theorem 3. *$\{A_0, \ldots, A_m\}$ is a Hausdorff moment sequence, if and only if the Hankel matrices satisfy the following inequalities:*

$$\mathbf{H}_n \geqq 0 \quad \textit{and} \quad \mathbf{K}_{n-1} \geqq \mathbf{L}_{n-1} \geqq 0 \qquad (\textit{if } m=2n),$$

or

$$\mathbf{H}_n \geqq \mathbf{K}_n \geqq 0 \qquad (\textit{if } m = 2n+1).$$

Proof. Suppose that $\{A_0, \ldots, A_m\}$ is a Hausdorff moment sequence. As in the proof of Theorem 1, there exist a selfadjoint operator $0 \leqq \hat{K} \leqq 1$ on a Hilbert space $\hat{\mathfrak{K}}$ and a bounded operator J from $\mathfrak{K}$ to $\hat{A}$ such that

$$(\mathbf{H}_n\langle f_0, \ldots, f_n\rangle, \langle f_0, \ldots, f_n\rangle) = \Big\|\sum_{k=0}^{n} \hat{A}^k \cdot Jf_k\Big\|^2$$

and

$$(\mathbf{K}_N\langle f_0, \ldots, f_N\rangle, \langle f_0, \ldots, f_N\rangle) = \Big(\hat{A}\Big(\sum_{k=0}^{N} \hat{A}^k \cdot Jf_k\Big), \sum_{k=0}^{N} \hat{A}^k \cdot Jf_k\Big),$$

where $N = n-1$ or $=n$ according as $m=2n$ or $= 2n+1$. Thus the Hankel matrix $\mathbf{H}_n$ is positive. The inequality $\mathbf{K}_{n-1} \geqq \mathbf{L}_{n-1}$ follows from the following relation, based on the property $0 \leqq \hat{A} \leqq 1$:

$$\Big(\hat{A}\Big(\sum_{k=0}^{n-1} \hat{A}^k \cdot Jf_k\Big), \sum_{k=0}^{n-1} \hat{A}^k \cdot Jf_k\Big) \geqq \Big\|\hat{A}\Big(\sum_{k=0}^{n-1} \hat{A}^k \cdot Jf_k\Big)\Big\|^2 =$$
$$= (\mathbf{L}_{n-1}\langle f_0, \ldots, f_{n-1}\rangle, \langle f_0, \ldots, f_{n-1}\rangle).$$

The inequality $\mathbf{H}_n \geqq \mathbf{K}_n$, in case $m = 2n+1$, follows from the relation:

$$\Big\|\sum_{k=0}^{n} \hat{A}^k \cdot Jf_k\Big\|^2 \geqq \Big(\hat{A}\Big(\sum_{k=0}^{n} \hat{A}^k \cdot Jf_k\Big), \sum_{k=0}^{n} \hat{A}^k \cdot Jf_k\Big).$$

Suppose, conversely, that $m=2n$, $\mathbf{H}_n \geqq 0$ and $\mathbf{K}_{n-1} \geqq \mathbf{L}_{n-1}$. Consider the shift operator $\mathbf{S}$ in $\bigoplus_0^n \mathfrak{K}$:

$$\mathbf{S}\langle f_0, \ldots, f_{n-1}, 0\rangle = \langle 0, f_0, \ldots, f_{n-1}\rangle.$$

Since

$$\|\mathbf{S}\langle f_0, \ldots, f_{n-1}, 0\rangle\|^2_{\mathbf{H}_n} =$$
$$= (\mathbf{L}_{n-1}\langle f_0, \ldots, f_{n-1}\rangle, \langle f_0, \ldots, f_{n-1}\rangle) \leqq (\mathbf{K}_{n-1}\langle f_0, \ldots, f_{n-1}\rangle, \langle f_0, \ldots, f_{n-1}\rangle) =$$
$$= (\mathbf{S}\langle f_0, \ldots, f_{n-1}, 0\rangle, \langle f_0, \ldots, f_{n-1}, 0\rangle)_{\mathbf{H}_n},$$

the operator $\mathbf{S}$ can be considered as a symmetric operator in $(\bigoplus_0^n \mathfrak{K})_{\mathbf{H}_n}$, and admits, by Corollary 2, a selfadjoint extension $0 \leqq \hat{\mathbf{S}} \leqq 1$ on $(\bigoplus_0^n \mathfrak{K})_{\mathbf{H}_n}$. Let $\mathbf{E}(\lambda)$ $(0 \leqq \lambda \leqq 1)$ be the resolution of identity for $\hat{\mathbf{S}}$, then just as in the proof of Theorem 1,

$\{A_0, \ldots, A_m\}$ is a Hausdorff moment sequence with respect to the spectral function $E(\lambda)$, defined by

$$(E(\lambda)f, g) = (\mathbf{E}(\lambda)\langle f, 0, \ldots, 0\rangle, \langle g, 0, \ldots, 0\rangle_{H_n}.$$

Suppose finally that $m = 2n+1$ and $\mathbf{H}_n \geqq \mathbf{K}_n \geqq 0$. Then the Hankel matrix $\mathbf{H}_{n+1}$ with $A_{2n+2} = A_{2n+1}$ is positive. In fact,

$$(\mathbf{H}_{n+1}\langle f_0, \ldots, f_{n+1}\rangle, \langle f_0, \ldots, f_{n+1}\rangle) \geqq (\mathbf{K}_n\langle f_0, \ldots, f_n\rangle, \langle f_0, \ldots, f_n\rangle) +$$
$$+ \sum_{k=0}^{n} (A_{n+1+k}f_k, f_{n+1}) + \sum_{k=0}^{n} (f_{n+1}, A_{n+1+k}f_k) + (A_{2n+1}f_{n+1}, f_{n+1}) =$$
$$= \|\langle f_0, \ldots, f_{n-1}, f_n + f_{n+1}\rangle\|^2_{\mathbf{K}_n}.$$

Consider the truncated shift operator $\mathbf{T}$ in $(\overset{n+1}{\underset{0}{\oplus}} \mathfrak{K})_{\mathbf{H}_{n+1}}$:

$$\mathbf{T}\langle f_0, \ldots, f_n, 0\rangle = \mathbf{P}\langle 0, f_0, \ldots, f_n\rangle,$$

where $\mathbf{P}$ is the orthogonal projection onto the closure of the canonical image of $\overset{n}{\underset{0}{\oplus}} \mathfrak{K}$. Since

$$\|\mathbf{T}\langle f_0, \ldots, f_n, 0\rangle\|^2_{\mathbf{H}_{n+1}} = \sup_{g \in \mathfrak{K}} \frac{|(\langle 0, f_0, \ldots, f_n\rangle, \langle g_0, \ldots, g_n, 0\rangle)_{\mathbf{H}_{n+1}}|^2}{\|\langle g_0, \ldots, g_n, 0\rangle\|^2_{\mathbf{H}_{n+1}}} =$$
$$= \sup_{g \in \mathfrak{K}} \frac{|(\mathbf{K}_n\langle f_0, \ldots, f_n\rangle, \langle g_0, \ldots, g_n\rangle)|^2}{(\mathbf{H}_n\langle g_0, \ldots, g_n\rangle, \langle g_0, \ldots, g_n\rangle)} \leqq (\mathbf{K}_n\langle f_0, \ldots, f_n\rangle, \langle f_0, \ldots, f_n\rangle) =$$
$$= (\mathbf{T}\langle f_0, \ldots, f_n, 0\rangle, \langle f_0, \ldots, f_n, 0\rangle)_{\mathbf{H}_{n+1}},$$

the operator $\mathbf{T}$ is considered as a symmetric operator in $(\overset{n+1}{\underset{0}{\oplus}} \mathfrak{K})_{\mathbf{H}_{n+1}}$ and admits, by Corollary 2, a selfadjoint extension $0 \leqq \hat{\mathbf{T}} \leqq 1$. It follows that for $j \leqq n+1$ and $k \leqq n$

$$(A_{j+k}f, g) = (\mathbf{S}^k\langle f, 0, \ldots, 0\rangle, \mathbf{S}^j\langle g, 0, \ldots, 0\rangle)_{\mathbf{H}_{n+1}} =$$
$$= (\mathbf{T}^k\langle f, 0, \ldots, 0\rangle, \mathbf{T}^j\langle g, 0, \ldots, 0\rangle)_{\mathbf{H}_{n+1}},$$

for

$$\mathbf{T}^k\langle f, 0, \ldots, 0\rangle = \mathbf{S}^k\langle f, 0, \ldots, 0\rangle \quad (k \leqq n)$$

and

$$\mathbf{S}^j\langle g, 0, \ldots, 0\rangle = \mathbf{S}\mathbf{T}^{j-1}\langle g, 0, \ldots, 0\rangle \qquad (1 \leqq j \leqq n+1).$$

Let $\mathbf{E}(\lambda)$ $(0 \leqq \lambda \leqq 1)$ be the resolution of identity for $\hat{\mathbf{T}}$. Then it follows that for $j \leqq 2n+1$

$$(A_j f, g) = (\hat{\mathbf{T}}^j\langle f, 0, \ldots, 0\rangle, \langle g, 0, \ldots, 0\rangle)_{\mathbf{H}_{n+1}} =$$
$$= \int_0^1 \lambda^j \, d(\mathbf{E}(\lambda)\langle f, 0, \ldots, 0\rangle, \langle g, 0, \ldots, 0\rangle)_{\mathbf{H}_{n+1}}.$$

Now $\{A_0, \ldots, A_{2n+1}\}$ is a Hausdorff moment sequence with respect to the spectral function $E(\lambda)$, defined by

$$(E(\lambda)f, g) = (\mathbf{E}(\lambda)\langle f, 0, \ldots, 0\rangle, \langle g, 0, \ldots, 0\rangle)_{\mathbf{H}_{n+1}}.$$

This completes the proof.

Remark 3. Kreĭn and Krasnoselskiĭ [5; § 7] proved Theorem 3 only for the case $m = 2n$.

6. Trigonometric truncated moment problem

A finite sequence $\{A_0, \ldots, A_n\}$ of bounded operators is called a *trigonometric moment sequence*, if there exists a spectral function $E(\lambda)$ $(-\pi \leqq \lambda \leqq \pi)$ such that

$$A_k = \int_{-\pi}^{\pi} e^{ik\lambda}\, dE(\lambda) \qquad (k = 0, 1, \ldots, n).$$

It is a *Kreĭn moment sequence*, if $E(\lambda)$ can be taken as a function on $[-\pi/2, \pi/2]$, i.e.

$$A_k = \int_{-\pi/2}^{\pi/2} e^{ik\lambda}\, dE(\lambda) \qquad (k = 0, 1, \ldots, n).$$

For a sequence $\{A_0, A_1, \ldots\}$, the *Toeplitz matrix* $\mathbf{T}_n$ and the *shifted Toeplitz matrix* $\mathbf{T}_{n,+}$ are the operators on $\bigoplus_0^n \mathfrak{K}$, given respectively by

$$\mathbf{T}_n = \begin{pmatrix} A_0, & A_1, & A_2, & \ldots, A_n \\ A_1^*, & A_0, & A_1, & \ldots, A_{n-1} \\ A_2^*, & A_1^*, & A_0, A_1, & \ldots, A_{n-2} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ A_n^*, & A_{n-1}^*, & A_{n-2}^*, & \ldots, A_0 \end{pmatrix} \quad \text{and} \quad \mathbf{T}_{n,+} = \begin{pmatrix} A_1, & A_2, & \ldots, A_{n+1} \\ A_0, & A_1, & \ldots, A_n \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ A_{n-1}^*, & A_{n-2}^*, & \ldots, A_1 \end{pmatrix}.$$

Theorem 4. *$\{A_0, \ldots, A_n\}$ is a trigonometric moment sequence, if and only if the Toeplitz matrix $\mathbf{T}_n$ is positive. It is a Kreĭn moment sequence, if and only if, in addition, the shifted Toeplitz matrix $\mathbf{T}_{n-1,+}$ has positive real part.*

Proof. Suppose that $\{A_0, \ldots, A_n\}$ is a trigonometric moment sequence. As in the proof of Theorem 1, there exists a unitary operator $\hat{U}$ on a Hilbert space $\hat{\mathfrak{K}}$ and a bounded operator J from $\mathfrak{K}$ to $\hat{\mathfrak{K}}$ such that

$$(\mathbf{T}_n\langle f_0, \ldots, f_n\rangle, \langle f_0, \ldots, f_n\rangle) = \left\| \sum_{k=0}^{n} \hat{U}^k \cdot Jf_k \right\|^2$$

and

$$(\mathbf{T}_{n-1,+}\langle f_0, \dots, f_{n-1}\rangle, \langle f_0, \dots, f_{n-1}\rangle) = \Big(\hat{U}\Big(\sum_{k=0}^{n-1} \hat{U}^k \cdot Jf_k\Big), \sum_{k=0}^{n-1} \hat{U}^k \cdot Jf_k\Big).$$

Thus the Toeplitz matrix $\mathbf{T}_n$ is positive. In case of a Kreĭn moment sequence, $\mathbf{T}_{n-1,+}$ has positive real part, for $\hat{U}$ has positive real part.

If, conversely, the Toeplitz matrix $\mathbf{T}_n$ is positive, the shift operator $\mathbf{V}$:

$$\mathbf{V}\langle f_0, \dots, f_{n-1}, 0\rangle = \langle 0, f_0, \dots, f_{n-1}\rangle$$

is isometric in $(\overset{n}{\underset{0}{\oplus}} \mathfrak{K})_{\mathbf{T}_n}$, hence there exists a spectral function $\mathbf{E}(\lambda)$ $(-\pi \leqq \lambda \leqq \pi)$ in $(\overset{n}{\underset{0}{\oplus}} \mathfrak{K})_{T_n}$ such that $\int_{-\pi}^{\pi} e^{ik\lambda} d\mathbf{E}(\lambda)$ is an extension of $\mathbf{V}^k$ for all k. It follows, as in the proof of Theorem 1,

$$A_k = \int_{-\pi}^{\pi} e^{ik\lambda}\, dE(\lambda) \qquad (k = 0, 1, \dots, n),$$

where $E(\lambda)$ is the spectral function in $\mathfrak{K}$, defined by

$$(E(\lambda)f, g) = (\mathbf{E}(\lambda)\langle f, 0, \dots, 0\rangle, \langle g, 0, \dots, 0\rangle)_{\mathbf{T}_n}.$$

If, in addition, the shifted Toeplitz matrix $\mathbf{T}_{n-1,+}$ has positive real part, it follows

$$\operatorname{Re}(\mathbf{V}\langle f_0, \dots, f_{n-1}, 0\rangle, \langle f_0, \dots, f_{n-1}, 0\rangle)_{\mathbf{T}_n} =$$
$$= (\operatorname{Re}(\mathbf{T}_{n-1,+})\langle f_0, \dots, f_{n-1}\rangle, \langle f_0, \dots, f_{n-1}\rangle) \geqq 0,$$

hence by Proposition 3 $\mathbf{E}(\lambda)$ can be taken as a function on $[-\pi/2, \pi/2]$. This completes the proof.

References

[1] N. I. Ahiezer, *The classical moment problem* (Edinburgh, 1965).
[2] N. I. Ahiezer and M. G. Kreĭn, *Some questions in the theory of moments* (Providence, 1962).
[3] T. Ando and K. Nishio, Selfadjoint extensions of positive symmetric operators, *Tohoku Math. J.*, **22** (1970), 65—75.
[4] M. A. Krasnoselskiĭ, On some types of extensions of Hermitian operators, *Ukrain. Math. J.*, **2** (1950) 74—83.
[5] M. G. Kreĭn and M. A. Krasnoselskiĭ, Fundamental theorems on the extension of Hermitian operators and some of their applications to the theory of orthogonal polynomials and the problems of moment, *Uspehi Mat. Nauk*, (N. S.) **2**: **3** (**19**) (1947), 60—106.
[6] F. Riesz and B. Sz.-Nagy, *Functional analysis* (New York, 1955).
[7] B. Sz.-Nagy, *Extensions of linear transformations in Hilbert space which extend beyond this space* (New York, 1960).

RESEARCH INSTITUTE OF APPLIED ELECTRICITY
HOKKAIDO UNIVERSITY, SAPPORO, JAPAN

(Received June 25, 1969)