Some characterizations of two-sided regular rings

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To Professor L. Rédei on his seventieth birthday

Throughout this paper by a ring we mean a not necessarily commutative but associative ring and by the radical of the ring we mean the Jacobson radical [6]. Following J. VON NEUMANN [14] we shall say that the ring A is *regular* if for every element a of A there exists an element x in A such that $axa=a$. It is well known that the class of regular rings plays a very important rôle in the abstract algebra, in the theory of Banach algebras (cf. C. E. RICKART [17]) and in the continuous geometry [15]. An interesting result is that the ring of all linear transformations of a vector space over a division ring is a regular ring. Some ideal-theoretical characterizations of regular rings were obtained by L. KOVÁCS [8] and J. LUH [12]. The regularity criterion of KOVÁCS reads as follows. An associative ring A is regular if and only if the relation

(1) $$R \cap L = RL$$

holds for every left ideal L and for every right ideal R of A.

Following E. HILLE [5] a ring A is called a *two-sided* ring if every one-sided (left or right) ideal of A is a two-sided ideal of A. Clearly every division ring is a two-sided ring, and so is every commutative ring. It is easy to see that there exists a two-sided ring which is neither commutative nor a division ring. Two-sided rings, called *duo rings,* were investigated by E. H. FELLER [3] and G. THIERRIN [22]. Thierrin proved, using the classical method of N. H. MCCOY [13], that every two-sided ring can be represented as a subdirect sum of subdirectly irreducible two-sided rings.

A. FORSYTHE and N. H. MCCOY [4] proved the assertion that a nonzero regular ring A is a subdirect sum of division rings if and only if the ring A does not contain nonzero nilpotent elements. Their proof uses among others the following lemmas: (1) If a nonzero idempotent element e of a subdirectly irreducible ring A lies in the center of A, then e is the identity element of A. (2) If a nonzero subdirectly irreducible regular ring does not contain nonzero nilpotent elements, then it is a division ring.

A ring A is called *strongly regular* (see R. F. ARENS and I. KAPLANSKY [2]) if to every element a of A there exists at least one element x of A such that $a=a^2x$. It can be seen that every strongly regular rings is regular (see T. KÁNDÔ [7]), and in a strongly regular ring $a=a^2x$ if and only if $a=xa^2$.

In a paper of the second author [18] it was proved that a ring with minimum condition on principal right ideals is a discrete direct sum of division rings if and only if the ring has no nonzero nilpotent elements. It is clear that this class of rings contains only regular two-sided rings.

The first named author has recently obtained ideal-theoretical characterizations of two-sided regular rings which are analogous to his characterizations of semilattices of groups [9], [10], [11]. His earlier criteria are also contained in the following result.

Theorem. *For an associative ring A the following conditions are mutually equivalent:*

(I) *A is a two-sided regular ring.*

(II) *$L\cap R = LR$ for every left ideal L and for every right ideal R of A.*

(III) *The intersection of any two left ideals is equal to their product and the same holds for right ideals too.*

(IV) *$L\cap I = LI$ and $R\cap I = IR$ for every left ideal L, for every right ideal R and, for every two-sided ideal I of A.*

(V) *A is regular and a subdirect sum of division rings.*

(VI) *A is a regular ring with no nonzero nilpotent elements.*

(VII) *A is strongly regular.*

(VIII) *The intersection of any two left ideals coincides with their product.*

(IX) *The intersection of any two right ideals coincides with their product.*

(X) *$L\cap I = LI$ holds for every left ideal L and for every two-sided ideal I of A.*

(XI) *$R\cap I = IR$ holds for every right ideal R and for every two-sided ideal I of A.*

Proof. (I)$\Rightarrow$(II). Let A be a two-sided regular rings. Then A satisfies the relation

$$L\cap R = RL \tag{2}$$

for every left ideal L and for every right ideal R of A by the regularity criterion of KOVÁCS. In case of two-sided rings this is equivalent to condition (II).

(II)$\Rightarrow$(I). Let A be an associative ring having the property (II). In the case of $R=A$ the condition (II) implies

$$A\cap L = LA, \tag{3}$$

that is, every left ideal L of A is also a right ideal of A. Similarly in case $L=A$ relation (II) implies

$$A\cap R = AR, \tag{4}$$

thus the right ideal R of A is a two-sided ideal of A. Therefore A is a two-sided ring. Finally (II) implies relation (2) which is equivalent to the regularity of A.

(I)⇔(III). The proof is similar to the above proof of the equivalence (I)⇔(II).

(I)⇒(IV). The proof is analogous to that of (I)⇒(II).

(IV)⇒(I). Let A be a ring with property (IV). In case $I=A$ we have

$$A \cap L = LA. \tag{5}$$

This means that any left ideal L of A is also a right ideal of A. Consequently the intersection of any two left ideals is equal to their product by (IV). Similarly it can be proved that every right ideal is a two-sided ideal of A and, the intersection of any two right ideals coincides with their product. Therefore (IV) implies (III), and we have already proved the implication (III)⇒(I), thus (IV) implies (I).

(I)⇒(V). Let A be an arbitrary regular two-sided ring. By the regularity of A the Jacobson radical J of A coincides with the ideal (0). Suppose that $J \neq (0)$. Then every nonzero principal right ideal of A contains a nonzero idempotent element e and the quasi-regularity condition

$$e+x-ex = 0 \tag{6}$$

multiplied on the left by e yields

$$e=0, \tag{7}$$

which is a contradiction to the supposition $e \neq 0$. Therefore we have $J=(0)$. Hence the intersection of all modular maximal right ideals I_α of A equals the ideal (0), that is

$$\bigcap_\alpha I_\alpha = (0). \tag{8}$$

Since A is a two-sided ring, every right ideal I_α is two-sided, hence the factor ring A/I_α has no nontrivial right ideals. By the modularity of I_α the factor ring A/I_α is a division ring and, the relation (8) implies the condition (V).

(V)⇒(VI). The proof is almost trivial, and we omit it.

(VI)⇒(VII). Let A be an arbitrary regular ring with no nonzero nilpotent elements. By the mentioned paper of FORSYTHE and McCOY every idempotent element of A belongs to the center of A. Suppose that $a=axa$ for $a \in A$, $x \in A$. Then the idempotent element $e=ax$ commutes with the element $a \in A$, therefore $a=a^2x$. Similarly the idempotent element $f=xa$ also commutes with a consequently $a=xa^2$, that is, A is strongly regular.

(VII)⇒(I). Let A be an arbitrary strongly regular ring. Then the relation $a \in a^2A$ for every $a \in A$ implies the fact that A has no nonzero nilpotent elements because in case $a^n=0$ one can conclude

$$a \in a^2A \subseteqq (a^3)_r \subseteqq a^2 \cdot a^2A \subseteqq (a^5)_r \subseteqq a^6A \subseteqq \ldots, \tag{9}$$

whence $a=0$.

We must yet prove that A is a two-sided ring. To this purpose it is enough to show that every principal right ideal of A is a two-sided ideal. By the regularity of A every principal right ideal of A can be generated by an idempotent element e of A. Let now a be an arbitrary element of the principal right ideal (e) generated by e. Then we have $e^2=e$ and $a=ea$ which imply

$$(10) \qquad (ar-a)^2 = 0.$$

Since A has no nonzero nilpotent element, we have $ae=a$ and hence $a\in(e)_l$ which implies the inclusion $(e)_l\subseteqq(e)_r$. The converse inclusion $(e)_l\subseteqq(e)_r$ can be proved similarly; consequently $(e)_r=(e)_l$, which means that A is indeed a two-sided regular ring. Therefore condition (I) holds.

(VII)$\Leftrightarrow$(IX). This result was proved by V. A. ANDRUNAKIEVIČ [1].

(VIII)$\Leftrightarrow$(IX). By a left-right duality and by the mentioned result of ANDRUNAKIEVIČ it is sufficient to prove that the condition $a\in Aa^2$ for every element $a\in A$ is equivalent to one of $a\in Aa^2$ for every element a of A. It was proved in the part (VII)$\Rightarrow$(I) that in the case $a\in a^2A$ $(\forall a\in A)$ the ring A has no nonzero nilpotent elements and, hence every idempotent element lies in the center of A by FORSYTHE and McCOY. Therefore $a=a^2x$ implies $a=axa$ and $a=xa^2$. The proof of the converse statement is similar.

(I)$\Rightarrow$(X). The proof is similar to that of (I)$\Rightarrow$(II).

(X)$\Rightarrow$(I). First in case $I=A$ condition (X) implies that every left ideal L of A is a two-sided ideal. Therefore assertion (X) implies (VIII), which is equivalent to (I).

(I)$\Rightarrow$(XI). The proof is the same as in the case (I)$\Rightarrow$(II).

(XI)$\Rightarrow$(I). The proof is similar to that of (X)$\Rightarrow$(I).

The proof of our Theorem is complete.

Remark 1. If the condition

$$(11) \qquad \bigcap_\alpha (R+I_\alpha) \subseteqq R+\bigcap_\alpha I_\alpha$$

holds for every right (and left) ideal R and for any system of two-sided ideals I_α of a ring A and A is a subdirect sum of division rings, then it can be proved by another method that A is a two-sided ring. Namely let us suppose that

$$(12) \qquad \bigcap_\alpha I_\alpha = (0)$$

holds for the two-sided ideals I_α of A, where the factor rings A/I_α are division rings. Then the images of the arbitrary right ideal R of A are two-sided ideals in the rings A/I_α. Furthermore the complete inverse images $R+I_\alpha$ of R are two-sided ideals in A by the first isomorphism theorem (see e.g. L. Rédei [16]). Then the condition (11) together with (12) implies

(13) $$\bigcap_{\alpha}(R+I_{\alpha}) \subseteqq R.$$

But conversely we trivially have

(14) $$R \subseteqq \bigcap_{\alpha}(R+I_{\alpha}),$$

whence

(15) $$R = \bigcap_{\alpha}(R+I_{\alpha}).$$

Here the intersection $\bigcap_{\alpha}(R+I_{\alpha})$ is a two-sided ideal of A, and thus R is also a two-sided ideal. Therefore A is a two-sided regular ring. Condition (11) seems to be very similar to the modularity condition of a lattice (see G. Szász [21]).

Remark 2. We mention a nontrivial example for a two-sided regular ring which is neither a commutative nor a division ring. Let A be the direct sum of two non-commutative division rings. Then A has obviously the wished properties.

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